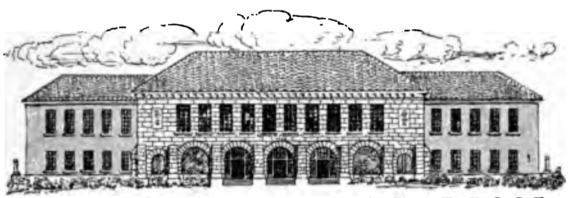
AND SOLID
ETRY WELLS



THE

ESSENTIALS OF GEOMETRY

BY

WEBSTER WELLS, S.B.

PROFESSOR OF MATHEMATICS IN THE MASSACHUSETTS
INSTITUTE OF TECHNOLOGY

BOSTON, U.S.A.
D. C. HEATH & CO., PUBLISHERS
1899

Norwood Press
J. S. Cushing & Co. – Berwick & Smith
Norwood Mass. U.S.A.

PREFACE.

In the *Essentials of Geometry*, the author has endeavored to prepare a work suited to the needs of high schools and academies. It will also be found to answer as well the requirements of colleges and scientific schools.

In some of its features, the work is similar to the author's *Revised Plane and Solid Geometry;* but important improvements have been introduced, which are in line with the present requirements of many progressive teachers.

In a number of propositions, the figure is given, and a statement of what is to be proved; the details of the proof being left to the pupil, usually with a hint as to the method of demonstration to be employed.

The propositions and corollaries left in this way for the pupil to demonstrate will be found in the following sections: —

Book I., §§ 51, 75, 76, 78, 79, 96, 102, 110, 111, 112, 115, 117, 136.

Book II., §§ 158, 160, 165, 170, 172 (Case III.), 174, 178, 179, 193 (Case III.), 194, and 201.

Book III., §§ 251, 257, 261, 264, 268, 278, 282, 284, and 286.

Book IV., §§ 312 and 316.

Book V., §§ 346, 347, and 350.

Book VI., §§ 405, 407, 412, 414, 415, 416, 417, 420, 421, 434, 437, 440, 442, and 444.

Book VII., §§ 491, 495, 507, 512, 513, 521, 528, 529, and 530.

Book VIII., §§ 554, 559, 578, 580, 581, 594, 595, 601, 603, 608, 613, 614, 625 (Case II.), 630, 631, 635, and 637.

Book IX., §§ 654, 656, 660, 673, and 679.

There are also Problems in Construction in which the construction or proof is left to the pupil.

Another important improvement consists in giving figures and suggestions for the exercises. In Book I., the pupil has a figure for every non-numerical exercise; after that, they are only given with the more difficult ones.

In many of the exercises in construction, the pupil is expected to discuss the problem, or point out its limitations.

In Book I., and also in the first eighteen propositions of Book VI., the authority for each statement of a proof is given directly after the statement, in smaller type, enclosed in brackets. In the remaining portions of the work, the formal statement of the authority is omitted; but the number of the section where it is to be found is usually given.

In a number of cases, however, where the pupil is presumed, from practice, to be so familiar with the authority as not to require reference to the section where it is to be found, there is given merely an interrogation-point.

In all these cases the pupil should be required to give the authority as carefully and accurately as if it were actually printed on the page.

Another improvement consists in marking the parts of a demonstration by the words *Given*, *To Prove*, and *Proof*, printed in heavy-faced type.

A similar system is followed in the Constructions, by the use of the words *Given*, *Required*, *Construction*, and *Proof*.

A minor improvement is the omission of the definite article in speaking of geometrical magnitudes; thus we speak of "angle *A*," "triangle *ABC*," etc., and not "the angle *A*," "the triangle *ABC*," etc.

Symbols and abbreviations have been freely used; a list of these will be found on page 4.

Particular attention has been given to putting the propositions in the first part of Book I. in a form adapted to the needs of a beginner.

The pages have been arranged in such a way as to avoid the necessity, while reading a proof, of turning the page for reference to the figure.

The Appendix to the Plane Geometry contains propositions on Maxima and Minima of Plane Figures, and Symmetrical Figures; also, additional exercises of somewhat greater difficulty than those previously given.

The Appendix to the Solid Geometry contains rigorous proofs of the limit statements made in §§ 639, 650, 667, and 674.

The author wishes to acknowledge, with thanks, the many suggestions which he has received from teachers in all parts of the country, which have added materially to the value of the work.

WEBSTER WELLS.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY,
1899.

Stereoscopic views of many of the figures in the Solid Geometry have been prepared. Full particulars may be obtained from the publishers.

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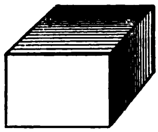
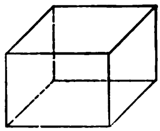
GEOMETRY.

PRELIMINARY DEFINITIONS.



A material body.



A geometrical solid.

1. A *material body*, such as a block of wood, occupies a *limited* or *bounded* portion of space.

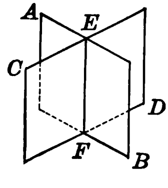
The boundary which separates such a body from surrounding space is called the *surface* of the body.

2. If the material composing such a body could be conceived as taken away from it, *without altering the form or shape of the bounding surface*, there would remain a *portion of space* having the same bounding surface as the former material body; this portion of space is called a *geometrical solid*, or simply a *solid*.

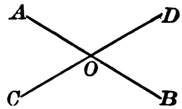
The surface which bounds it is called a *geometrical surface*, or simply a *surface;* it is also called the *surface of the solid*.

3. If two geometrical surfaces intersect each other, that which is common to both is called a *geometrical line*, or simply a *line*.

Thus, if surfaces *AB* and *CD* cut each other, their common intersection, *EF*, is a line.



4. If two geometrical lines intersect each other, that which is common to both is called a *geometrical point*, or simply a *point*.



Thus, if lines *AB* and *CD* cut each other, their common intersection, *O*, is a point.

5. A solid has *extension in every direction;* but this is not true of surfaces and lines.

A point has extension in *no* direction, but simply *position in space.*

6. A surface may be conceived as existing independently in space, without reference to the solid whose boundary it forms.

In like manner, we may conceive of lines and points as having an independent existence in space.

7. A *straight line,* or *right line,* is a line which has the same direction throughout its length; as *AB.*

A straight line.

A curve.

A broken line.

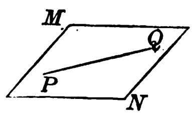
A *curved line,* or *curve,* is a line no portion of which is straight; as *CD.*

A *broken line* is a line which is composed of different successive straight lines; as *EFGH.*

8. The word "*line*" will be used hereafter as signifying a *straight line.*

9. A *plane surface,* or *plane,* is a surface such that the straight line joining any two of its points lies entirely in the surface.

Thus, if *P* and *Q* are any two points in surface *MN,* and the straight line joining *P* and *Q* lies entirely in the surface, then *MN* is a plane.



10. A *curved surface* is a surface no portion of which is plane.

11. We may conceive of a straight line as being of unlimited extent in regard to length; and in like manner we may conceive of a plane as being of unlimited extent in regard to length and breadth.

12. A *geometrical figure* is any combination of points, lines, surfaces, or solids.

A *plane figure* is a figure formed by points and lines all lying in the same plane.

A geometrical figure is called *rectilinear*, or *right-lined*, when it is composed of straight lines only.

13. *Geometry* treats of the properties, construction, and measurement of geometrical figures.

14. *Plane Geometry* treats of plane figures only.

Solid Geometry, also called *Geometry of Space*, or *Geometry of Three Dimensions*, treats of figures which are not plane.

15. An *Axiom* is a truth which is assumed without proof as being self-evident.

A *Theorem* is a truth which requires demonstration.

A *Problem* is a question proposed for solution.

A *Proposition* is a general term for a theorem or problem.

A *Postulate* assumes the possibility of solving a certain problem.

A *Corollary* is a secondary theorem, which is an immediate consequence of the proposition which it follows.

A *Scholium* is a remark or note.

An *Hypothesis* is a supposition made either in the statement or the demonstration of a proposition.

16. Postulates.

1. We assume that a straight line can be drawn between any two points.

2. We assume that a straight line can be produced (*i.e.*, prolonged) indefinitely in either direction.

17. Axioms.

We assume the truth of the following:

1. *Things which are equal to the same thing, or to equals, are equal to each other.*

2. *If the same operation be performed upon equals, the results will be equal.*

3. *But one straight line can be drawn between two points.*

4. *A straight line is the shortest line between two points.*

5. *The whole is equal to the sum of all its parts.*

6. *The whole is greater than any of its parts.*

18. Since but one straight line can be drawn between two points, a straight line is said to be *determined* by any two of its points.

19. Symbols and Abbreviations.

The following symbols will be used in the work:

$+$, plus.
$-$, minus.
$\times$, multiplied by.
$=$, equals.
$\backsimeq$, equivalent, is equivalent to.
$>$, is greater than.
$<$, is less than.
$\therefore$, therefore.
$\angle$, angle.
$\angle$s, angles.

$\triangle$, triangle.
$\triangle$s, triangles.
$\perp$, perpendicular, is perpendicular to.
$\perp$s, perpendiculars.
$\parallel$, parallel, is parallel to.
$\parallel$s, parallels.
▱, parallelogram.
▱s, parallelograms.
$\odot$, circle.
$\odot$s, circles.

The following abbreviations will also be used:

Ax.,	Axiom.	Sup.,	Supplementary.
Def.,	Definition.	Alt.,	Alternate.
Hyp.,	Hypothesis.	Int.,	Interior.
Cons.,	Construction.	Ext.,	Exterior.
Rt.,	Right.	Corresp.,	Corresponding.
Str.,	Straight.	Rect.,	Rectangle, rectangular.
Adj.,	Adjacent.		

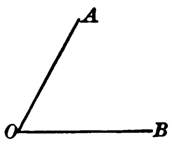
PLANE GEOMETRY.

Book I.

RECTILINEAR FIGURES.

DEFINITIONS AND GENERAL PRINCIPLES.

20. An *angle* (∠) is the *amount of divergence* of two straight lines which are drawn from the same point in different directions.



The point is called the *vertex* of the angle, and the straight lines are called its *sides.*

21. If there is but one angle at a given vertex, it may be designated by the letter at that vertex; but if two or more angles have the same vertex, we avoid ambiguity by naming also a letter on each side, placing the letter at the vertex between the others.

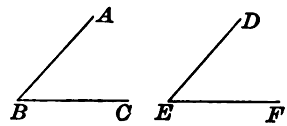
Thus, we should call the angle of § 20 "angle O"; but if there were other angles having the same vertex, we should read it either AOB or BOA.

Another way of designating an angle is by means of a letter placed between its sides; examples of this will be found in § 71.

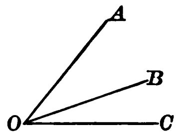
22. Two geometrical figures are said to be *equal* when one can be applied to the other so that they shall coincide throughout.

To prove two *angles* equal, we do not consider the lengths of their sides.

Thus, if angle ABC can be applied to angle DEF in such a manner that point B shall fall on point E, and sides AB and BC on sides DE and EF, respectively, the angles are equal, even if sides AB and BC are not equal in length to sides DE and EF, respectively.



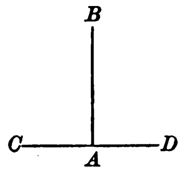
23. Two angles are said to be *adjacent* when they have the same vertex, and a common side between them; as AOB and BOC.



PERPENDICULAR LINES.

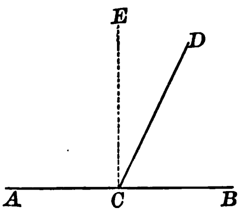
24. If from a given point in a straight line a line be drawn meeting the given line in such a way as to make the adjacent angles equal, each of the equal angles is called a *right angle*, and the lines are said to be *perpendicular* (⊥) to each other.

Thus, if from point A in straight line CD line AB be drawn in such a way as to make angles BAC and BAD equal, each of these angles is a right angle, and AB and CD are perpendicular to each other.



PROP. I. THEOREM.

25. *At a given point in a straight line, a perpendicular to the line can be drawn, and but one.*



Let C be the given point in straight line AB.

To prove that a perpendicular can be drawn to *AB* at *C*, and but one.

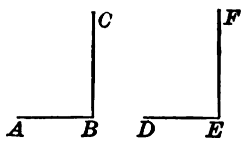
Draw a straight line *CD* in such a position that angle *BCD* shall be less than angle *ACD*; and let line *CD* be turned about point *C* as a pivot towards the position *CA*.

Then, angle *BCD* will constantly increase; and angle *ACD* will constantly diminish, until it becomes less than angle *BCD*; and it is evident that there is one position of *CD*, and only one, in which these angles are equal.

Let *CE* be this position; then by the definition of § 24, *CE* is perpendicular to *AB*.

Hence, a perpendicular can be drawn to *AB* at *C*, and but one.

26. Cor. *All right angles are equal.*



Let *ABC* and *DEF* be right angles.

To prove angles *ABC* and *DEF* equal.

Let angle *ABC* be superposed (*i.e.*, placed) upon angle *DEF* in such a way that point *B* shall fall upon point *E*, and line *AB* upon line *DE*.

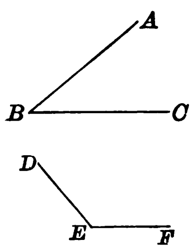
Then, line *BC* will fall upon line *EF*; for otherwise we should have two lines perpendicular to *DE* at *E*, which is impossible.

[At a given point in a straight line, but one perpendicular to the line can be drawn.] (§ 25)

Hence, angles *ABC* and *DEF* are equal (§ 22).

DEFINITIONS.

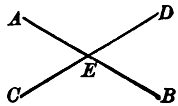
27. An *acute* angle is an angle which is less than a right angle; as *ABC*.



An *obtuse* angle is an angle which is greater than a right angle; as *DEF*.

Acute and obtuse angles are called *oblique* angles; and intersecting lines which are not perpendicular, are said to be oblique to each other.

28. Two angles are said to be *vertical*, or *opposite*, when the sides of one are the prolongations of the sides of the other; as *AEC* and *BED*.



29. An angle is *measured* by finding how many times it contains another angle, adopted arbitrarily as the unit of measure.

The usual unit of measure is the *degree*, which is the ninetieth part of a right angle.

To express fractional parts of the unit, the degree is divided into sixty equal parts called *minutes*, and the minute into sixty equal parts, called *seconds*.

Degrees, minutes, and seconds are represented by the symbols, °, ′, ″, respectively.

Thus, 43° 22′ 37″ represents an angle of 43 degrees, 22 minutes, and 37 seconds.

30. If the sum of two angles is a right angle, or 90°, one is called the *complement* of the other; and if their sum is two right angles, or 180°, one is called the *supplement* of the other.

For example, the complement of an angle of 34° is 90° − 34°, or 56°; and the supplement of an angle of 34° is 180° − 34°, or 146°.

Two angles which are complements of each other are called *complementary;* and two angles which are supplements of each other are called *supplementary.*

31. It is evident that

1. *The complements of equal angles are equal.*
2. *The supplements of equal angles are equal.*

EXERCISES.

1. How many degrees are there in the complement of 47°? of 83°? of 90°?

2. How many degrees are there in the supplement of 31°? of 90°? of 178°?

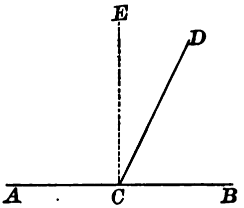
3. How many degrees are there in the complement, and in the supplement, of an angle equal to $\frac{7}{12}$ of a right angle?

4. How many degrees are there in an angle whose supplement is equal to $\frac{26}{11}$ of its complement?

5. Two angles are complementary, and the greater exceeds the less by 37°. How many degrees are there in each angle?

Prop. II. Theorem.

32. *If two adjacent angles have their exterior sides in the same straight line, their sum is equal to two right angles.*



Let angles ACD and BCD have their sides AC and BC in the same straight line.

To prove the sum of angles ACD and BCD equal to two right angles.

Draw line CE perpendicular to AB at C.

[At a given point in a straight line, a perpendicular to the line can be drawn.] (§ 25)

Then, it is evident that the sum of angles ACD and BCD is equal to the sum of angles ACE and BCE.

But since CE is perpendicular to AB, angles ACE and BCE are right angles.

Hence, the sum of angles ACD and BCD is equal to two right angles.

33. Sch. Since angles ACD and BCD are *supplementary* (§ 30), the theorem may be stated as follows:

If two adjacent angles have their exterior sides in the same straight line, they are supplementary.

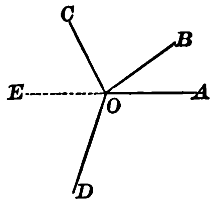
Such angles are called *supplementary-adjacent.*

34. Cor. I. *The sum of all the angles on the same side of a straight line at a given point is equal to two right angles.*

This is evident from § 32.

35. Cor. II. *The sum of all the angles about a point in a plane is equal to four right angles.*

Let *AOB, BOC, COD,* and *DOA* be angles about the point *O*.



To prove the sum of angles *AOB, BOC, COD,* and *DOA* equal to four right angles.

Produce *AO* to *E*.

Then, the sum of angles *AOB, BOC,* and *COE* is equal to two right angles.

[The sum of all the angles on the same side of a straight line at a given point is equal to two right angles.] (§ 34)

In like manner, the sum of angles *EOD* and *DOA* is equal to two right angles.

Therefore, the sum of angles *AOB, BOC, COD,* and *DOA* is equal to four right angles.

Ex. 6. If, in the figure of § 35, angles *AOB, BOC,* and *COD* are respectively 49°, 88°, and $\frac{8}{9}$ of a right angle, how many degrees are there in angle *AOD*?

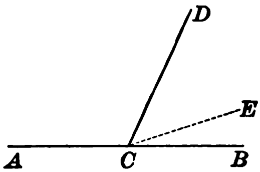
36. Sch. The pupil will now observe that a demonstration, in Geometry, consists of three parts:

1. *The statement of what is given in the figure.*
2. *The statement of what is to be proved.*
3. *The proof.*

In the remaining propositions of the work, we shall mark clearly the three divisions of the demonstration by heavy-faced type, and employ the symbols and abbreviations of § 20.

Prop. III. Theorem.

37. *If the sum of two adjacent angles is equal to two right angles, their exterior sides lie in the same straight line.*



Given the sum of adj. ∠s ACD and BCD equal to two rt. ∠s.

To Prove that AC and BC lie in the same str. line.

Proof. If AC and BC do not lie in the same str. line, let CE be in the same str. line with AC.

Then since ACE is a str. line, $\angle ECD$ is the supplement of $\angle ACD$.

[If two adj. ∠s have their ext. sides in the same str. line, they are supplementary.] (§ 33)

But by hyp., $\angle ACD + \angle BCD =$ two rt. ∠s.

Whence, $\angle BCD$ is the supplement of $\angle ACD$. (§ 30)

Then since both $\angle ECD$ and $\angle BCD$ are supplements of $\angle ACD$, $\angle ECD = \angle BCD$.

[The supplements of equal ∠s are equal.] (§ 31)

Hence, EC coincides with BC, and AC and BC lie in the same str. line.

38. Sch. I. It will be observed that the enunciation of every theorem consists essentially of two parts; the *Hypothesis*, and the *Conclusion*.

Thus, we may enunciate Prop. I as follows:

Hypothesis. If a point be taken in a given straight line,

Conclusion. A perpendicular to the line at the given point can be drawn, and but one.

39. Sch. II. We may enunciate Prop. II as follows:

Hypothesis. If two adjacent angles have their exterior sides in the same straight line,

Conclusion. Their sum is equal to two right angles.

Again, we may enunciate Prop. III:

Hypothesis. If the sum of two adjacent angles is equal to two right angles,

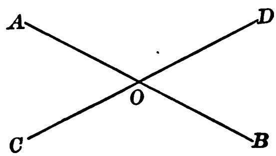
Conclusion. Their exterior sides lie in the same straight line.

One proposition is said to be the **Converse** of another when the hypothesis and conclusion of the first are, respectively, the conclusion and hypothesis of the second.

It is evident from the above considerations that Prop. III is the *converse* of Prop. II.

PROP. IV. THEOREM.

40. *If two straight lines intersect, the vertical angles are equal.*



Given str. lines AB and CD intersecting at O.

To Prove $\angle AOC = \angle BOD$.

Proof. Since $\angle s\ AOC$ and AOD have their ext. sides in str. line CD, $\angle AOC$ is the supplement of $\angle AOD$.

[If two adj. $\angle s$ have their ext. sides in the same str. line, they are supplementary.] (§ 33)

For the same reason, $\angle BOD$ is the supplement of $\angle AOD$.

$$\therefore \angle AOC = \angle BOD.$$

[The supplements of equal $\angle s$ are equal.] (§ 31)

In like manner, we may prove

$$\angle AOD = \angle BOC.$$

EXERCISES.

7. If, in the figure of Prop. IV., $\angle AOD = 137°$, how many degrees are there in BOC? in AOC? in BOD?

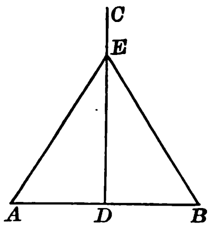
8. Two angles are supplementary, and the greater is seven times the less. How many degrees are there in each angle?

PROP. V. THEOREM.

41. *If a perpendicular be erected at the middle point of a straight line,*

I. *Any point in the perpendicular is equally distant from the extremities of the line.*

II. *Any point without the perpendicular is unequally distant from the extremities of the line.*



I. **Given** line $CD \perp$ to line AB at its middle point D, E any point in CD, and lines AE and BE.

To Prove $AE = BE$.

Proof. Superpose figure BDE upon figure ADE by folding it over about line DE as an axis.

Now $\angle BDE = \angle ADE$.

[All rt. $\angle$s are equal.] (§ 26)

Then, line BD will fall upon line AD.

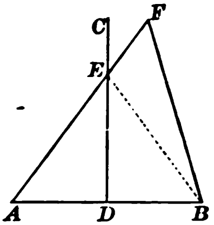
But by hyp., $BD = AD$.

Whence, point B will fall on point A.

Then line BE will coincide with line AE.

[But one str. line can be drawn between two points.] (Ax. 3)

$\therefore AE = BE$.



II. **Given** line $CD \perp$ to line AB at its middle point D, F any point without CD, and lines AF and BF.

To Prove $AF > BF$.

Proof. Let AF intersect CD at E, and draw line BE.

Now $BE + EF > BF$.

[A str. line is the shortest line between two points.] (Ax. 4)

But, $BE = AE$.

[If a $\perp$ be erected at the middle point of a str. line, any point in the $\perp$ is equally distant from the extremities of the line.] (§ 41, I)

Substituting for BE its equal AE, we have

$$AE + EF > BF, \text{ or } AF > BF.$$

42 Cor. I. *Every point which is equally distant from the extremities of a straight line, lies in the perpendicular erected at the middle point of the line.*

43. Cor. II. Since a straight line is determined by any two of its points (§ 18), it follows from § 42 that

Two points, each equally distant from the extremities of a straight line, determine a perpendicular at its middle point.

44. Cor. III. When figure BDE is superposed upon figure ADE, in the proof of § 41, I., $\angle EBD$ coincides with $\angle EAD$, and $\angle BED$ with $\angle AED$.

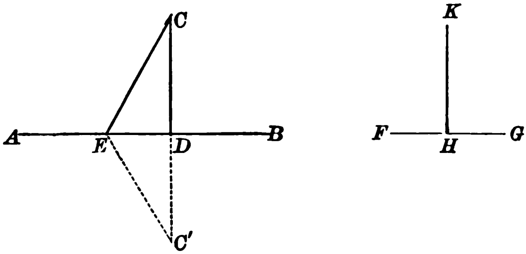
That is, $\angle EAD = \angle EBD$, and $\angle AED = \angle BED$.

Then, *if lines be drawn to the extremities of a straight line from any point in the perpendicular erected at its middle point,*

1. *They make equal angles with the line.*
2. *They make equal angles with the perpendicular.*

Prop. VI. Theorem.

45. *From a given point without a straight line, a perpendicular can be drawn to the line, and but one.*



Given point C without line AB.

To Prove that a $\perp$ can be drawn from C to AB, and but one.

Proof. Let line HK be $\perp$ to line FG at H.

[At a given point in a str. line, a $\perp$ to the line can be drawn.] (§ 25)

Apply line FG to line AB, and move it along until HK passes through C; let point H fall at D, and draw line CD.

Then, CD is $\perp$ AB.

If possible, let CE be another $\perp$ from C to AB.

Produce CD to C', making $C'D = CD$, and draw line EC'.

By cons., ED is $\perp$ to CC' at its middle point D.

$$\therefore \angle CED = \angle C'ED.$$

[If lines be drawn to the extremities of a str. line from any point in the $\perp$ erected at its middle point, they make equal $\angle$s with the $\perp$.] (§ 44)

But by hyp., $\angle CED$ is a rt. $\angle$; then, $\angle C'ED$ is a rt. $\angle$.

$$\therefore \angle CED + \angle C'ED = \text{two rt. } \angle\text{s}.$$

Then line CEC' is a str. line.

[If the sum of two adj. $\angle$s is equal to two rt. $\angle$s, their ext. sides lie in the same str. line.] (§ 37)

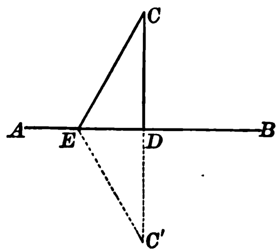
But this is impossible, for, by cons., CDC' is a str. line.

[But one str. line can be drawn between two points.] (Ax. 3)

Hence, CE cannot be $\perp$ AB, and CD is the only $\perp$ that can be drawn.

Prop. VII. Theorem.

46. *The perpendicular is the shortest line that can be drawn from a point to a straight line.*



Given CD the $\perp$ from point C to line AB, and CE any other str. line from C to AB.

To Prove $CD < CE$.

Proof. Produce CD to C', making $C'D = CD$, and draw line EC'.

By cons., ED is $\perp$ to CC' at its middle point D.

$$\therefore CE = C'E.$$

[If a $\perp$ be erected at the middle point of a str. line, any point in the $\perp$ is equally distant from the extremities of the line.] (§ 41)

But $$CD + DC' < CE + EC'.$$

[A str. line is the shortest line between two points.] (Ax 4.)

Substituting for DC' and EC' their equals CD and CE, respectively, we have

$$2\,CD < 2\,CE.$$

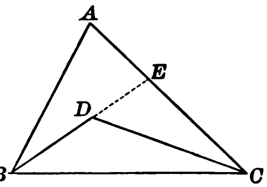
$$\therefore CD < CE.$$

47. Sch. The *distance* of a point from a line is understood to mean the length of the perpendicular from the point to the line.

Ex. 9. Find the number of degrees in the angle the sum of whose supplement and complement is 196°.

Prop. VIII. Theorem.

48. *If two lines be drawn from a point to the extremities of a straight line, their sum is greater than the sum of two other lines similarly drawn, but enveloped by them.*



Given lines AB and AC drawn from point A to the extremities of line BC; and DB and DC two other lines similarly drawn, but enveloped by AB and AC.

To Prove $AB + AC > DB + DC.$

Proof. Produce BD to meet AC at E.

Now $AB + AE > BE.$

[A str. line is the shortest line between two points.] (Ax. 4)

Adding EC to both members of the inequality,

$$BA + AC > BE + EC.$$

Again, $DE + EC > DC.$

Adding BD to both members of the inequality,

$$BE + EC > BD + DC.$$

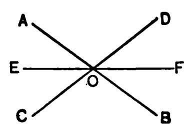
Since $BA + AC$ is greater than $BE + EC$, which is itself greater than $BD + DC$, it follows that

$$AB + AC > DB + DC.$$

EXERCISES.

10. The straight line which bisects an angle bisects also its vertical angle.

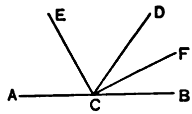
(If OE bisects $\angle AOC$, $\angle AOE = \angle COE$; and these ∠s are equal to ∠s BOF and DOF, respectively.)



11. The bisectors of a pair of vertical angles lie in the same straight line.

(Fig. of Ex. 10. To prove *EOF* a str. line. $\angle COE = \angle DOF$, for they are the halves of equal $\angle$s; but $\angle DOE + \angle COE = 2$ rt. $\angle$s, and therefore $\angle DOE + \angle DOF = 2$ rt. $\angle$s.)

12. The bisectors of two supplementary adjacent angles are perpendicular to each other.

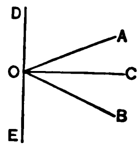


(We have $\angle ACD + \angle BCD = 2$ rt. $\angle$s; and $\angle$s *DCE* and *DCF* are the halves of $\angle$s *ACD* and *BCD*, respectively.)

13. If the bisectors of two adjacent angles are perpendicular, the angles are supplementary.

(Fig. of Ex. 12. Sum of $\angle$s *DCE* and $DCF = 1$ rt. $\angle$, and $\angle$s *DCE* and *DCF* are the halves of $\angle$s *ACD* and *BCD*, respectively.)

14. A line drawn through the vertex of an angle perpendicular to its bisector makes equal angles with the sides of the given angle.



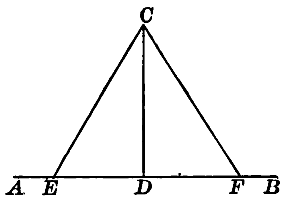
($\angle$s *AOD* and *BOE* are complements of $\angle$s *AOC* and *BOC*, respectively.)

Prop. IX. Theorem.

49. *If oblique lines be drawn from a point to a straight line,*

I. *Two oblique lines cutting off equal distances from the foot of the perpendicular from the point to the line are equal.*

II. *Of two oblique lines cutting off unequal distances from the foot of the perpendicular from the point to the line, the more remote is the greater.*



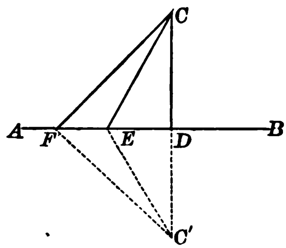
I. **Given** *CD* the $\perp$ from point *C* to line *AB*; and *CE* and *CF* oblique lines from *C* to *AB*, cutting off equal distances from the foot of *CD*.

To Prove $CE = CF.$

Proof. By hyp., CD is $\perp$ to EF at its middle point D.

$$\therefore\ CE = CF.$$

[If a $\perp$ be erected at the middle point of a str. line, any point in the $\perp$ is equally distant from the extremities of the line.] (§ 41)



II. **Given** CD the $\perp$ from point C to line AB; and CE and CF oblique lines from C to AB, cutting off unequal distances from the foot of CD; CF being the more remote.

To Prove $CF > CE.$

Proof. Produce CD to C', making $C'D = CD$, and draw lines $C'E$ and $C'F$.

By cons., AD is $\perp$ to CC' at its middle point D.

$$\therefore\ CF = C'F, \text{ and } CE = C'E.$$

[If a $\perp$ be erected at the middle point of a str. line, any point in the $\perp$ is equally distant from the extremities of the line.] (§ 41)

But $CF + FC' > CE + EC'.$

[If two lines be drawn from a point to the extremities of a str. line, their sum is $>$ the sum of two other lines similarly drawn, but enveloped by them.] (§ 48)

Substituting for FC' and EC' their equals CF and CE, respectively, we have

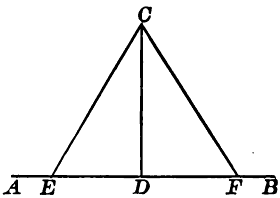
$$2\,CF > 2\,CE.$$

$$\therefore\ CF > CE.$$

Note. The theorem holds equally if oblique line CE is on the opposite side of perpendicular CD from CF.

Prop. X. Theorem.

50. (Converse of Prop. IX., I.) *If oblique lines be drawn from a point to a straight line, two equal oblique lines cut off equal distances from the foot of the perpendicular from the point to the line.*



Given CD the $\perp$ from point C to line AB, and CE and CF equal oblique lines from C to AB.

To Prove $DE = DF$.

Proof. We know that DE is either $>$, equal to, or $<$ DF.

If we suppose $DE > DF$, CE would be $> CF$.

[If oblique lines be drawn from a point to a str. line, of two oblique lines cutting off unequal distances from the foot of the $\perp$ from the point to the line, the more remote is the greater.] (§ 49)

But this is contrary to the hypothesis that $CE = CF$.

Hence, DE cannot be $> DF$.

In like manner, if we suppose $DE < DF$, CE would be $< CF$, which is contrary to the hypothesis that $CE = CF$.

Hence, DE cannot be $< DF$.

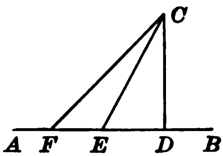
Then, if DE can be neither $> DF$, nor $< DF$, we must have

$$DE = DF.$$

Note. The method of proof exemplified in Prop. X is known as the "Indirect Method," or the "*Reductio ad Absurdum*."

The truth of a proposition is demonstrated by making every possible supposition in regard to the matter, and showing that, in all cases except the one which we wish to prove, the supposition leads to something which is contrary to the hypothesis.

51. Cor. (Converse of Prop. IX, II.) *If two unequal oblique lines be drawn from a point to a straight line, the greater cuts off the greater distance from the foot of the perpendicular from the point to the line.*



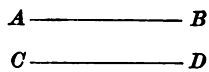
Given CD the ⊥ from point C to line AB; and CE and CF unequal oblique lines from C to AB, CF being $> CE$.

To Prove $DF > DE$.

(Prove by *Reductio ad Absurdum;* by § 49, I, DE cannot equal DF, and by § 49, II, it cannot be $> DF$.)

PARALLEL LINES.

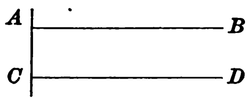
52. Def. Two straight lines are said to be *parallel* (‖) when they lie in the same plane, and cannot meet however far they may be produced; as AB and CD.



53. Ax. *We assume that but one straight line can be drawn through a given point parallel to a given straight line.*

PROP. XI. THEOREM.

54. *Two perpendiculars to the same straight line are parallel.*



Given lines AB and CD ⊥ to line AC.

To Prove $AB \parallel CD$.

Proof. If AB and CD are not ‖, they will meet in some point if sufficiently produced (§ 52).

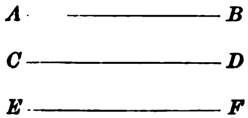
We should then have two ⊥s from this point to AC, which is impossible.

[From a given point without a str. line, but one ⊥ can be drawn to the line.] (§ 45)

Therefore, AB and CD cannot meet, and are ‖.

PROP. XII. THEOREM.

55. *Two straight lines parallel to the same straight line are parallel to each other.*



Given lines AB and CD ∥ to line EF.

To Prove $AB \parallel CD$.

Proof. If AB and CD are not ∥, they will meet in some point if sufficiently produced. (§ 52)

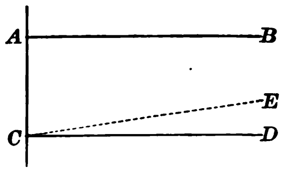
We should then have two lines drawn through this point ∥ to EF, which is impossible.

[But one str. line can be drawn through a given point ∥ to a given str. line.] (§ 53)

Therefore, AB and CD cannot meet, and are ∥.

PROP. XIII. THEOREM.

56. *A straight line perpendicular to one of two parallels is perpendicular to the other.*



Given lines AB and CD ∥, and line $AC \perp AB$.

To Prove $AC \perp CD$.

Proof. If CD is not $\perp AC$, let line CE be $\perp AC$.

Then since AB and CE are $\perp AC$, $CE \parallel AB$.

[Two ⊥s to the same str. line are ∥.] (§ 54)

But by hyp., $CD \parallel AB$.

Then, CE must coincide with CD.

[But one str. line can be drawn through a given point ∥ to a given str. line.] (§ 53)

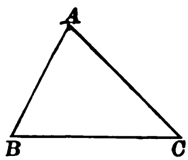
But by cons., $AC \perp CE$.

Then since CE coincides with CD, we have $AC \perp CD$.

TRIANGLES.

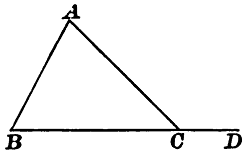
DEFINITIONS.

57. A *triangle* (△) is a portion of a plane bounded by three straight lines; as ABC.



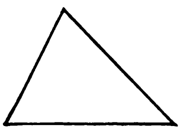
The bounding lines, AB, BC, and CA, are called the *sides* of the triangle, and their points of intersection, A, B, and C, the *vertices*.

The *angles* of the triangle are the angles CAB, ABC, and BCA, included between the adjacent sides.

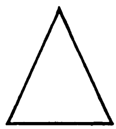


An *exterior angle* of a triangle is the angle at any vertex between any side of the triangle and the adjacent side produced; as ACD.

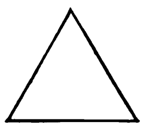
58. A triangle is called *scalene* when no two of its sides are equal; *isosceles* when two of its sides are equal; *equilateral* when all its sides are equal; and *equiangular* when all its angles are equal.



Scalene.

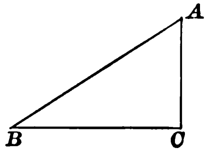


Isosceles.

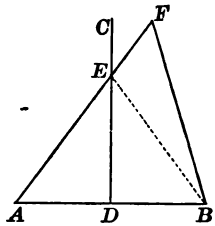


Equilateral.

59. A *right triangle* is a triangle which has a right angle; as ABC, which has a right angle at C.



The side AB opposite the right angle is called the *hypotenuse*, and the other sides, AC and BC, the *legs*.



II. **Given** line $CD \perp$ to line AB at its middle point D, F any point without CD, and lines AF and BF.

To Prove $AF > BF$.

Proof. Let AF intersect CD at E, and draw line BE.

Now $BE + EF > BF$.

[A str. line is the shortest line between two points.] (Ax. 4)

But, $BE = AE$.

[If a $\perp$ be erected at the middle point of a str. line, any point in the $\perp$ is equally distant from the extremities of the line.] (§ 41, I)

Substituting for BE its equal AE, we have

$$AE + EF > BF, \text{ or } AF > BF.$$

42. Cor. I. *Every point which is equally distant from the extremities of a straight line, lies in the perpendicular erected at the middle point of the line.*

43. Cor. II. Since a straight line is determined by any two of its points (§ 18), it follows from § 42 that

Two points, each equally distant from the extremities of a straight line, determine a perpendicular at its middle point.

44. Cor. III. When figure BDE is superposed upon figure ADE, in the proof of § 41, I., $\angle EBD$ coincides with $\angle EAD$, and $\angle BED$ with $\angle AED$.

That is, $\angle EAD = \angle EBD$, and $\angle AED = \angle BED$.

Then, *if lines be drawn to the extremities of a straight line from any point in the perpendicular erected at its middle point,*

1. *They make equal angles with the line.*
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Prop. VI. Theorem.

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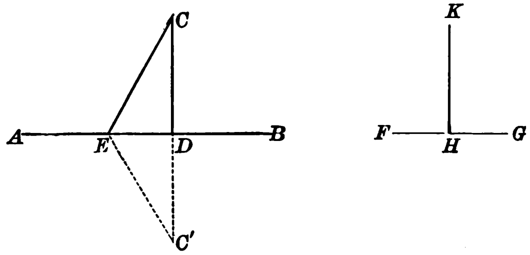

Given point C without line AB.

To Prove that a $\perp$ can be drawn from C to AB, and but one.

Proof. Let line HK be $\perp$ to line FG at H.

[At a given point in a str. line, a $\perp$ to the line can be drawn.] (§ 25)

Apply line FG to line AB, and move it along until HK passes through C; let point H fall at D, and draw line CD.

Then, CD is $\perp$ AB.

If possible, let CE be another $\perp$ from C to AB.

Produce CD to C', making $C'D = CD$, and draw line EC'.

By cons., ED is $\perp$ to CC' at its middle point D.

$$\therefore \angle CED = \angle C'ED.$$

[If lines be drawn to the extremities of a str. line from any point in the $\perp$ erected at its middle point, they make equal $\angle$s with the $\perp$.] (§ 44)

But by hyp., $\angle CED$ is a rt. $\angle$; then, $\angle C'ED$ is a rt. $\angle$.

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Then line CEC' is a str. line.

[If the sum of two adj. $\angle$s is equal to two rt. $\angle$s, their ext. sides lie in the same str. line.] (§ 37)

But this is impossible, for, by cons., CDC' is a str. line.

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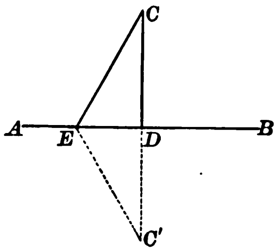

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To Prove $CD < CE$.

Proof. Produce CD to C', making $C'D = CD$, and draw line EC'.

By cons., ED is $\perp$ to CC' at its middle point D.

$$\therefore CE = C'E.$$

[If a $\perp$ be erected at the middle point of a str. line, any point in the $\perp$ is equally distant from the extremities of the line.] (§ 41)

But $CD + DC' < CE + EC'$.

[A str. line is the shortest line between two points.] (Ax 4.)

Substituting for DC' and EC' their equals CD and CE, respectively, we have

$$2\,CD < 2\,CE.$$

$$\therefore CD < CE.$$

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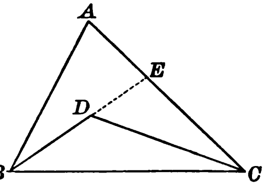

Given lines AB and AC drawn from point A to the extremities of line BC; and DB and DC two other lines similarly drawn, but enveloped by AB and AC.

To Prove $AB + AC > DB + DC$.

Proof. Produce BD to meet AC at E.

Now $AB + AE > BE$.

[A str. line is the shortest line between two points.] (Ax. 4)

Adding EC to both members of the inequality,

$$BA + AC > BE + EC.$$

Again, $DE + EC > DC$.

Adding BD to both members of the inequality,

$$BE + EC > BD + DC.$$

Since $BA + AC$ is greater than $BE + EC$, which is itself greater than $BD + DC$, it follows that

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EXERCISES.

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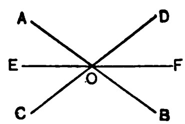
(If OE bisects $\angle AOC$, $\angle AOE = \angle COE$; and these $\angle$s are equal to $\angle$s BOF and DOF, respectively.)



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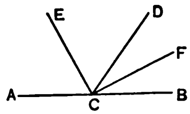

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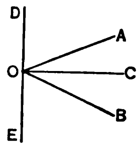
13. If the bisectors of two adjacent angles are perpendicular, the angles are supplementary.

(Fig. of Ex. 12. Sum of ∠s DCE and $DCF = 1$ rt. ∠, and ∠s DCE and DCF are the halves of ∠s ACD and BCD, respectively.)

14. A line drawn through the vertex of an angle perpendicular to its bisector makes equal angles with the sides of the given angle.



(∠s AOD and BOE are complements of ∠s AOC and BOC, respectively.)

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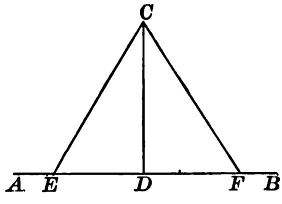
I. *Two oblique lines cutting off equal distances from the foot of the perpendicular from the point to the line are equal.*

II. *Of two oblique lines cutting off unequal distances from the foot of the perpendicular from the point to the line, the more remote is the greater.*



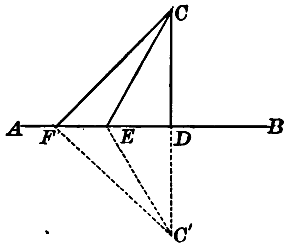
I. **Given** CD the ⊥ from point C to line AB; and CE and CF oblique lines from C to AB, cutting off equal distances from the foot of CD.

To Prove $CE = CF$.

Proof. By hyp., CD is ⊥ to EF at its middle point D.

$$\therefore\ CE = CF.$$

[If a ⊥ be erected at the middle point of a str. line, any point in the ⊥ is equally distant from the extremities of the line.] (§ 41)



II. **Given** CD the ⊥ from point C to line AB; and CE and CF oblique lines from C to AB, cutting off unequal distances from the foot of CD; CF being the more remote.

To Prove $CF > CE$.

Proof. Produce CD to C', making $C'D = CD$, and draw lines $C'E$ and $C'F$.

By cons., AD is ⊥ to CC' at its middle point D.

$$\therefore\ CF = C'F, \text{ and } CE = C'E.$$

[If a ⊥ be erected at the middle point of a str. line, any point in the ⊥ is equally distant from the extremities of the line.] (§ 41)

But $CF + FC' > CE + EC'$.

[If two lines be drawn from a point to the extremities of a str. line, their sum is > the sum of two other lines similarly drawn, but enveloped by them.] (§ 48)

Substituting for FC' and EC' their equals CF and CE, respectively, we have

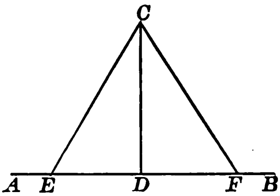
$$2\,CF > 2\,CE.$$

$$\therefore\ CF > CE.$$

Note. The theorem holds equally if oblique line CE is on the opposite side of perpendicular CD from CF.

Prop. X. Theorem.

50. (Converse of Prop. IX., I.) *If oblique lines be drawn from a point to a straight line, two equal oblique lines cut off equal distances from the foot of the perpendicular from the point to the line.*



Given CD the ⊥ from point C to line AB, and CE and CF equal oblique lines from C to AB.

To Prove $DE = DF$.

Proof. We know that DE is either $>$, equal to, or $<$ DF.

If we suppose $DE > DF$, CE would be $> CF$.

[If oblique lines be drawn from a point to a str. line, of two oblique lines cutting off unequal distances from the foot of the ⊥ from the point to the line, the more remote is the greater.] (§ 49)

But this is contrary to the hypothesis that $CE = CF$.

Hence, DE cannot be $> DF$.

In like manner, if we suppose $DE < DF$, CE would be $< CF$, which is contrary to the hypothesis that $CE = CF$.

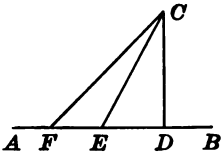
Hence, DE cannot be $< DF$.

Then, if DE can be neither $> DF$, nor $< DF$, we must have $DE = DF$.

Note. The method of proof exemplified in Prop. X is known as the "Indirect Method," or the "*Reductio ad Absurdum*."

The truth of a proposition is demonstrated by making every possible supposition in regard to the matter, and showing that, in all cases except the one which we wish to prove, the supposition leads to something which is contrary to the hypothesis.

51. Cor. (Converse of Prop. IX, II.) *If two unequal oblique lines be drawn from a point to a straight line, the greater cuts off the greater distance from the foot of the perpendicular from the point to the line.*



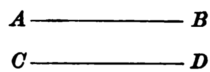
Given CD the ⊥ from point C to line AB; and CE and CF unequal oblique lines from C to AB, CF being $> CE$.

To Prove $DF > DE$.

(Prove by *Reductio ad Absurdum;* by § 49, I, DE cannot equal DF, and by § 49, II, it cannot be $> DF$.)

PARALLEL LINES.

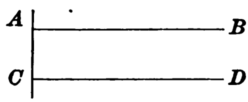
52. Def. Two straight lines are said to be *parallel* (∥) when they lie in the same plane, and cannot meet however far they may be produced; as AB and CD.



53. Ax. *We assume that but one straight line can be drawn through a given point parallel to a given straight line.*

Prop. XI. Theorem.

54. *Two perpendiculars to the same straight line are parallel.*



Given lines AB and CD ⊥ to line AC.

To Prove $AB \parallel CD$.

Proof. If AB and CD are not ∥, they will meet in some point if sufficiently produced (§ 52).

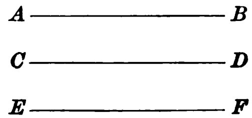
We should then have two ⊥s from this point to AC, which is impossible.

[From a given point without a str. line, but one ⊥ can be drawn to the line.] (§ 45)

Therefore, AB and CD cannot meet, and are ∥.

PROP. XII. THEOREM.

55. *Two straight lines parallel to the same straight line are parallel to each other.*



Given lines AB and CD ∥ to line EF.

To Prove $AB \parallel CD$.

Proof. If AB and CD are not ∥, they will meet in some point if sufficiently produced. (§ 52)

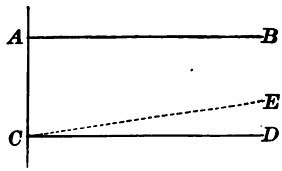
We should then have two lines drawn through this point ∥ to EF, which is impossible.

[But one str. line can be drawn through a given point ∥ to a given str. line.] (§ 53)

Therefore, AB and CD cannot meet, and are ∥.

PROP. XIII. THEOREM.

56. *A straight line perpendicular to one of two parallels is perpendicular to the other.*



Given lines AB and CD ∥, and line $AC \perp AB$.

To Prove $AC \perp CD$.

Proof. If CD is not ⊥ AC, let line CE be ⊥ AC.

Then since AB and CE are ⊥ AC, $CE \parallel AB$.

[Two ⊥s to the same str. line are ∥.] (§ 54)

But by hyp., $CD \parallel AB$.

Then, CE must coincide with CD.

[But one str. line can be drawn through a given point ∥ to a given str. line.] (§ 53)

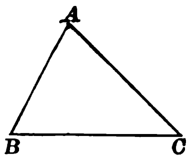
But by cons., $AC \perp CE.$

Then since CE coincides with CD, we have $AC \perp CD$.

TRIANGLES.

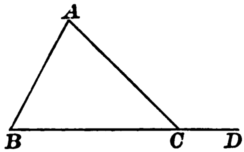
DEFINITIONS.

57. A *triangle* ($\triangle$) is a portion of a plane bounded by three straight lines; as ABC.



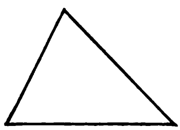
The bounding lines, AB, BC, and CA, are called the *sides* of the triangle, and their points of intersection, A, B, and C, the *vertices*.

The *angles* of the triangle are the angles CAB, ABC, and BCA, included between the adjacent sides.

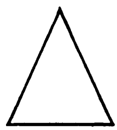


An *exterior angle* of a triangle is the angle at any vertex between any side of the triangle and the adjacent side produced; as ACD.

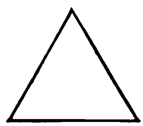
58. A triangle is called *scalene* when no two of its sides are equal; *isosceles* when two of its sides are equal; *equilateral* when all its sides are equal; and *equiangular* when all its angles are equal.



Scalene.

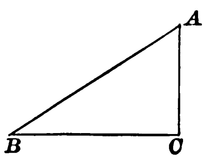


Isosceles.



Equilateral.

59. A *right triangle* is a triangle which has a right angle; as ABC, which has a right angle at C.

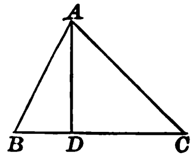


The side AB opposite the right angle is called the *hypotenuse*, and the other sides, AC and BC, the *legs*.

60. If any side of a triangle be taken and called the *base*, the corresponding *altitude* is the perpendicular drawn from the opposite vertex to the base, produced if necessary.

In general, either side may be taken as the base; but in an isosceles triangle, unless otherwise specified, the side which is not one of the equal sides is taken as the base.

When any side has been taken as the base, the opposite angle is called the *vertical angle*, and its vertex is called the *vertex of the triangle*.



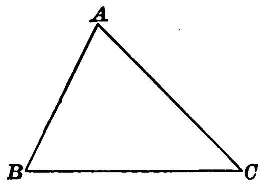
Thus, in triangle ABC, BC is the base, AD the altitude, and BAC the vertical angle.

61. Since a straight line is the shortest line between two points (Ax. 4), it follows that

Any side of a triangle is less than the sum of the other two sides.

PROP. XIV. THEOREM.

62. *Any side of a triangle is greater than the difference of the other two sides.*



Given AB, any side of $\triangle ABC$; and side $BC >$ side AC.

To Prove $AB > BC - AC$.

Proof. We have $AB + AC > BC$.

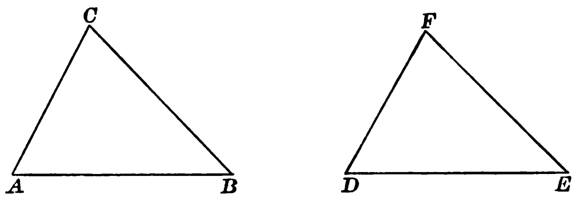
[A str. line is the shortest line between two points.] (Ax. 4)

Subtracting AC from both members of the inequality,

$$AB > BC - AC.$$

PROP. XV. THEOREM.

63. *Two triangles are equal when two sides and the included angle of one are equal respectively to two sides and the included angle of the other.*



Given, in $\triangle$s ABC and DEF,

$$AB = DE,\ AC = DF, \text{ and } \angle A = \angle D.$$

To Prove $\triangle ABC = \triangle DEF$.

Proof. Superpose $\triangle ABC$ upon $\triangle DEF$ in such a way that $\angle A$ shall coincide with its equal $\angle D$; side AB falling on side DE, and side AC on side DF.

Then since $AB = DE$ and $AC = DF$, point B will fall on point E, and point C on point F.

Whence, side BC will coincide with side EF.

[But one str. line can be drawn between two points.] (Ax. 3)

Therefore, the $\triangle$s coincide throughout, and are equal.

64. Cor. Since ABC and DEF coincide throughout, we have $\angle B = \angle E$, $\angle C = \angle F$, and $BC = EF$.

65. Sch. I. In equal figures, lines or angles which are similarly placed are called *homologous*.

Thus, in the figure of Prop. XV, $\angle A$ is homologous to $\angle D$; AB is homologous to DE; etc.

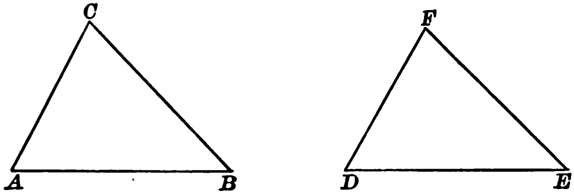
66. Sch. II. It follows from § 65 that

In equal figures, the homologous parts are equal.

67. Sch. III. In equal triangles, the equal angles lie opposite the equal sides.

Prop. XVI. Theorem.

68. *Two triangles are equal when a side and two adjacent angles of one are equal respectively to a side and two adjacent angles of the other.*



Given, in $\triangle$ ABC and DEF,

$$AB = DE,\ \angle A = \angle D, \text{ and } \angle B = \angle E.$$

To Prove $\triangle ABC = \triangle DEF$.

Proof. Superpose $\triangle ABC$ upon $\triangle DEF$ in such a way that side AB shall coincide with its equal DE; point A falling on point D, and point B on point E.

Then since $\angle A = \angle D$, side AC will fall on side DF, and point C will fall somewhere on DF.

And since $\angle B = \angle E$, side BC will fall on side EF, and point C will fall somewhere on EF.

Then point C, falling at the same time on DF and EF, must fall at their intersection, F.

Therefore, the $\triangle$ coincide throughout, and are equal.

EXERCISES.

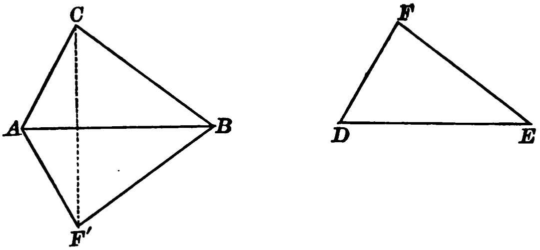
15. If, in the figure of Prop. XV., $AB = EF$, $BC = DE$, and $\angle B = \angle E$, which angle of triangle DEF is equal to A? which angle is equal to C?

16. If, in the figure of Prop. XVI., $AC = DF$, $\angle A = \angle F$, and $\angle C = \angle D$, which side of triangle DEF is equal to AB? which side is equal to BC?

17. If OD and OE are the bisectors of two complementary-adjacent angles, AOB and BOC, how many degrees are there in $\angle DOE$?

Prop. XVII. Theorem.

69. *Two triangles are equal when the three sides of one are equal respectively to the three sides of the other.*



Given, in ▲ ABC and DEF,

$$AB = DE,\ BC = EF, \text{ and } CA = FD.$$

To Prove $\triangle ABC = \triangle DEF$.

Proof. Place $\triangle DEF$ in the position ABF'; side DE coinciding with its equal AB, and vertex F falling at F', on the opposite side of AB from C.

Draw line CF'.

By hyp., $AC = AF'$ and $BC = BF'$.

Whence, AB is ⊥ to CF' at its middle point.

[Two points, each equally distant from the extremities of a str. line, determine a ⊥ at its middle point.] (§ 43)

$$\therefore \angle BAC = \angle BAF'.$$

[If lines be drawn to the extremities of a str. line from any point in the ⊥ erected at its middle point, they make equal ∠s with the ⊥.] (§ 44)

Then since sides AB and AC and $\angle BAC$ of $\triangle ABC$ are equal, respectively, to sides AB and AF' and $\angle BAF'$ of $\triangle ABF'$,

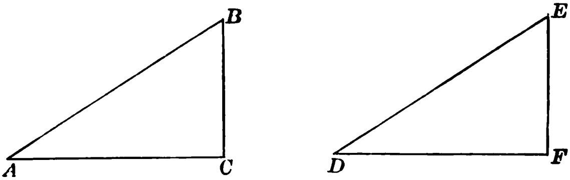
$$\triangle ABC = \triangle ABF'.$$

[Two ▲ are equal when two sides and the included ∠ of one are equal respectively to two sides and the included ∠ of the other.] (§ 63)

That is, $\triangle ABC = \triangle DEF$.

PROP. XVIII. THEOREM.

70. *Two right triangles are equal when the hypotenuse and an adjacent angle of one are equal respectively to the hypotenuse and an adjacent angle of the other.*



Given, in rt. ⊿ ABC and DEF,

hypotenuse AB = hypotenuse DE, and $\angle A = \angle D$.

To Prove $\triangle ABC = \triangle DEF$.

Proof. Superpose $\triangle ABC$ upon $\triangle DEF$ in such a way that hypotenuse AB shall coincide with its equal DE; point A falling on point D, and point B on point E.

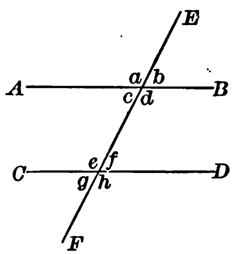
Then since $\angle A = \angle D$, side AC will fall on side DF.

Therefore, side BC will fall on side EF.

[From a given point without a str. line, but one ⊥ can be drawn to the line.] (§ 45)

Therefore, the ⊿ coincide throughout, and are equal.

71. Def. If two straight lines, AB and CD, are cut by a line EF, called a *transversal*, the angles are named as follows:



c, d, e, and f are called *interior* angles, and a, b, g, and h *exterior* angles.

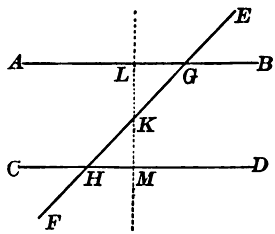
c and f, or d and e, are called *alternate-interior* angles.

a and h, or b and g, are called *alternate-exterior* angles.

a and e, b and f, c and g, or d and h, are called *corresponding* angles.

PROP. XIX. THEOREM.

72. *If two parallels are cut by a transversal, the alternate-interior angles are equal.*



Given ‖s AB and CD cut by transversal EF at points G and H, respectively.

To Prove $\angle AGH = \angle GHD$ and $\angle BGH = \angle CHG$.

Proof. Through K, the middle point of GH, draw line $LM \perp AB$; then, $LM \perp CD$.

[A str. line ⊥ to one of two ‖s is ⊥ to the other.] (§ 56)

Now in rt. △s GKL and HKM, by cons.,

$$\text{hypotenuse } GK = \text{hypotenuse } HK.$$

Also, $$\angle GKL = \angle HKM.$$

[If two str. lines intersect, the vertical ∠s are equal.] (§ 40)

$$\therefore \triangle GKL = \triangle HKM.$$

[Two rt. △s are equal when the hypotenuse and an adj. ∠ of one are equal respectively to the hypotenuse and an adj. ∠ of the other.] (§ 70)

$$\therefore \angle KGL = \angle KHM.$$

[In equal figures, the homologous parts are equal.] (§ 66)

Again, $\angle KGL$ is the supplement of $\angle BGH$, and $\angle KHM$ the supplement of $\angle CHG$.

[If two adj. ∠s have their ext. sides in the same str. line, they are supplementary.] (§ 33)

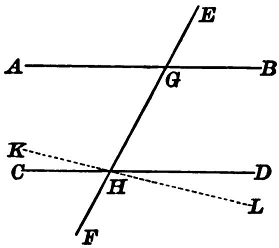
Then since $\angle KGL = \angle KHM$, we have

$$\angle BGH = \angle CHG.$$

[The supplements of equal ∠s are equal.] (§ 31)

PROP. XX. THEOREM.

73. (Converse of Prop. XIX.) *If two straight lines are cut by a transversal, and the alternate-interior angles are equal, the two lines are parallel.*



Given lines AB and CD cut by transversal EF at points G and H, respectively, and

$$\angle AGH = \angle GHD.$$

To Prove $AB \parallel CD.$

Proof. If CD is not $\parallel$ AB, draw line KL through $H \parallel AB$.
Then since $\parallel$s AB and KL are cut by transversal EF,

$$\angle AGH = \angle GHL.$$

[If two $\parallel$s are cut by a transversal, the alt. int. $\angle$s are equal.] (§ 72)

But by hyp., $\angle AGH = \angle GHD.$

$$\therefore \angle GHL = \angle GHD.$$

[Things which are equal to the same thing are equal to each other.] (Ax. 1)

But this is impossible unless KL coincides with CD.

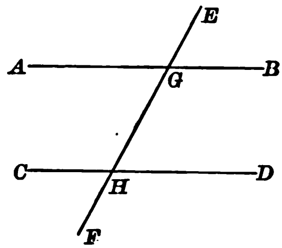
$$\therefore CD \parallel AB.$$

In like manner, it may be proved that if AB and CD are cut by EF, and $\angle BGH = \angle CHG$, then $AB \parallel CD$.

Ex. 18. If, in the figure of Prop. XIX., $\angle AGH = 68°$, how many degrees are there in BGH? in GHD? in DHF?

PROP. XXI. THEOREM.

74. *If two parallels are cut by a transversal, the corresponding angles are equal.*



Given ∥s AB and CD cut by transversal EF at points G and H, respectively.

To Prove $\angle AGE = \angle CHG$.

Proof. We have $\angle BGH = \angle CHG$.

[If two ∥s are cut by a transversal, the alt. int. ∠s are equal.] (§ 72)

But, $\angle BGH = \angle AGE$.

[If two str. lines intersect, the vertical ∠s are equal.] (§ 40)

$\therefore \angle AGE = \angle CHG$.

[Things which are equal to the same thing are equal to each other.] (Ax. 1)

In like manner, we may prove

$\angle AGH = \angle CHF$, $\angle BGE = \angle DHG$, and $\angle BGH = \angle DHF$.

75. Cor. I. *If two parallels are cut by a transversal, the alternate-exterior angles are equal.*

(Fig. of Prop. XXI.)

Given ∥s AB and CD cut by transversal EF at points G and H, respectively.

To Prove $\angle AGE = \angle DHF$.

($\angle BGH = \angle CHG$, and the theorem follows by § 40.)

What other two ext. ∠s in the figure are equal?

76. Cor. II. *If two parallels are cut by a transversal, the sum of the interior angles on the same side of the transversal is equal to two right angles.*

(Fig. of Prop. XXI.)

Given ‖s AB and CD cut by transversal EF at points G and H, respectively.

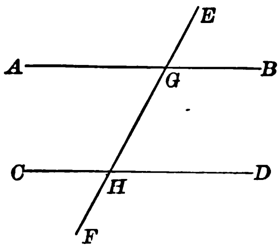
To Prove $\angle AGH + \angle CHG =$ two rt. $\angle$s.

(By § 32, $\angle AGH + \angle AGE =$ two rt. $\angle$s; the theorem follows by § 74.)

What other two int. $\angle$s in the figure have their sum equal to two rt. $\angle$s?

Prop. XXII. Theorem.

77. (Converse of Prop. XXI.) *If two straight lines are cut by a transversal, and the corresponding angles are equal, the two lines are parallel.*



Given lines AB and CD cut by transversal EF at points G and H, respectively, and

$$\angle AGE = \angle CHG.$$

To Prove $AB \parallel CD$.

Proof. We have $\angle AGE = \angle BGH$.

[If two str. lines intersect, the vertical $\angle$s are equal.] (§ 40)

$$\therefore \angle BGH = \angle CHG.$$

[Things which are equal to the same thing are equal to each other.] (Ax. 1)

$\therefore AB \parallel CD.$

[If two str. lines are cut by a transversal, and the alt. int. ∠s are equal, the two lines are ∥.] (§ 73)

In like manner, it may be proved that if

$\angle AGH = \angle CHF$, or $\angle BGE = \angle DHG$, or $\angle BGH = \angle DHF$,

then $AB \parallel CD$.

78. Cor. I. (Converse of § 75.) *If two straight lines are cut by a transversal, and the alternate-exterior angles are equal, the two lines are parallel.*

(Fig. of Prop. XXII.)

Given lines AB and CD cut by transversal EF at points G and H, respectively, and

$$\angle AGE = \angle DHF.$$

To Prove $AB \parallel CD$.

($\angle AGE = \angle BGH$, and $\angle DHF = \angle CHG$; and the theorem follows by § 73.)

What other two ext. ∠s are there in the figure such that, if they are equal, $AB \parallel CD$?

79. Cor. II. (Converse of § 76.) *If two straight lines are cut by a transversal, and the sum of the interior angles on the same side of the transversal is equal to two right angles, the two lines are parallel.*

(Fig. of Prop. XXII.)

Given lines AB and CD cut by transversal EF at points G and H, respectively, and

$$\angle AGH + \angle CHG = \text{two rt. } \angle s.$$

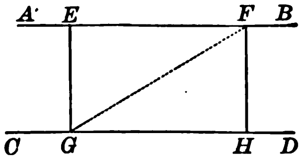
To Prove $AB \parallel CD$.

($\angle CHG$ is the supplement of $\angle AGH$, and also of $\angle GHD$; then ∠s AGH and GHD are equal by § 31, 2, and the theorem follows by § 73.)

What other two int. ∠s are there in the figure such that, if their sum equals two rt. ∠s, $AB \parallel CD$?

PROP. XXIII. THEOREM.

80. *Two parallel lines are everywhere equally distant.*



Given ∥s AB and CD, E and F any two points on AB, and EG and FH lines ⊥ CD.

To Prove $EG = FH$ (§ 47).

Proof. Draw line FG.

We have $EG \perp AB$.

[A str. line ⊥ to one of two ∥s is ⊥ to the other.] (§ 56)

Then, in rt. △s EFG and FGH,

$$FG = FG.$$

And since ∥s AB and CD are cut by FG,

$$\angle EFG = \angle FGH.$$

[If two ∥s are cut by a transversal, the alt. int. ∠s are equal.] (§ 72)

$$\therefore \triangle EFG = \triangle FGH.$$

[Two rt. △s are equal when the hypotenuse and an adj. ∠ of one are equal respectively to the hypotenuse and an adj. ∠ of the other.] (§ 70)

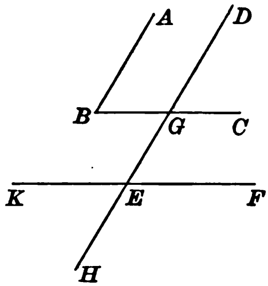
$$\therefore EG = FH.$$

[In equal figures, the homologous parts are equal.] (§ 66)

PROP. XXIV. THEOREM.

81. *Two angles whose sides are parallel, each to each, are equal if both pairs of parallel sides extend in the same direction, or in opposite directions, from their vertices.*

Note. The sides extend in the *same* direction if they are on the *same* side of a straight line joining the vertices, and in *opposite* directions if they are on *opposite* sides of this line.



Given lines AB and BC ∥ to lines DH and KF, respectively, intersecting at E.

I. **To Prove** that ∠s ABC and DEF, whose sides AB and DE, and also BC and EF, extend in the same direction from their vertices, are equal.

Proof. Let BC and DH intersect at G.

Since ∥s AB and DE are cut by BC,

$$\angle ABC = \angle DGC.$$

[If two ∥s are cut by a transversal, the corresp. ∠s are equal.] (§ 74)

In like manner, since ∥s BC and EF are cut by DE,

$$\angle DGC = \angle DEF.$$

$$\therefore \angle ABC = \angle DEF. \quad (1)$$

[Things which are equal to the same thing are equal to each other.] (Ax. 1)

II. **To Prove** that ∠s ABC and HEK, whose sides AB and EH, and also BC and EK, extend in opposite directions from their vertices, are equal.

Proof. From (1), $\angle ABC = \angle DEF$.

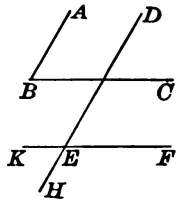
But, $\angle DEF = \angle HEK$.

[If two str. lines intersect, the vertical ∠s are equal.] (§ 40)

$$\therefore \angle ABC = \angle HEK.$$

[Things which are equal to the same thing, are equal to each other.] (Ax. 1)

82. Cor. *Two angles whose sides are parallel, each to each, are supplementary if one pair of parallel sides extend in the same direction, and the other pair in opposite directions, from their vertices.*



Given lines *AB* and *BC* ∥ to lines *DH* and *KF*, respectively, intersecting at *E*.

To Prove that ∠s *ABC* and *DEK*, whose sides *AB* and *DE* extend in the same direction, and *BC* and *EK* in opposite directions, from their vertices, are supplementary.

Proof. We have $\angle ABC = \angle DEF$.

[Two ∠s whose sides are ∥, each to each, are equal if both pairs of ∥ sides extend in the same direction from their vertices.] (§ 81)

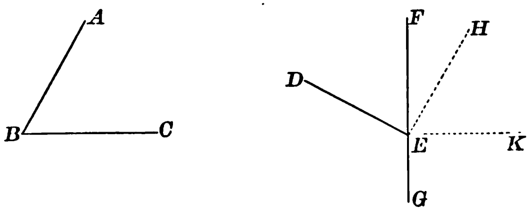
But ∠ *DEF* is the supplement of ∠ *DEK*.

[If two adj. ∠s have their ext. sides in the same str. line, they are supplementary.] (§ 33)

Then its equal, ∠ *ABC*, is the supplement of ∠ *DEK*.

Prop. XXV. Theorem.

83. *Two angles whose sides are perpendicular, each to each, are either equal or supplementary.*



Given lines *AB* and *BC* ⊥ to lines *DE* and *FG*, respectively, intersecting at *E*.

To Prove ∠ *ABC* equal to ∠ *DEF*, and supplementary to ∠ *DEG*.

Proof. Draw line $EH \perp DE$, and line $EK \perp EF$.

Then since EH and AB are $\perp DE$,

$$EH \parallel AB.$$

[Two ⊥s to the same str. line are ∥.] (§ 54)

In like manner, since EK and BC are $\perp EF$,

$$EK \parallel BC.$$

$$\therefore \angle HEK = \angle ABC.$$

[Two ∠s whose sides are ∥, each to each, are equal if both pairs of ∥ sides extend in the same direction from their vertices.] (§ 81)

But since, by cons., ∠s DEH and FEK are rt. ∠s, each of the ∠s DEF and HEK is the complement of $\angle FEH$.

$$\therefore \angle DEF = \angle HEK.$$

[The complements of equal ∠s are equal.] (§ 31)

$$\therefore \angle ABC = \angle DEF.$$

[Things which are equal to the same thing are equal to each other.] (Ax. 1)

Again, $\angle DEF$ is the supplement of $\angle DEG$.

[If two adj. ∠s have their ext. sides in the same str. line, they are supplementary.] (§ 33)

Then, its equal, $\angle ABC$ is the supplement of $\angle DEG$.

Note. The angles are equal if they are both acute or both obtuse; and supplementary if one is acute and the other obtuse.

EXERCISES.

19. If, in the figure of Prop. XXIV., $\angle ABC = 59°$, how many degrees are there in each of the angles formed about the point E?

20. The line passing through the vertex of an angle perpendicular to its bisector bisects the supplementary adjacent angle.

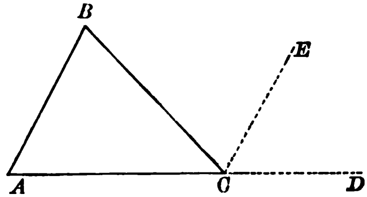
(Fig. of Ex. 12. Let CE bisect $\angle ACD$, and suppose $CF \perp CE$; sum of ∠s ACD and $BCD = 2$ rt. ∠s; then sum of ∠s DCE and $\frac{1}{2} BCD = 1$ rt. ∠; but sum of ∠s DCE and DCF is also 1 rt. ∠; whence the theorem follows.)

21. Any side of a triangle is less than the half-sum of the sides of the triangle.

(Fig. of Prop. XIV. We have $AB < BC + CA$; then add AB to both members of the inequality.)

PROP. XXVI. THEOREM.

84. *The sum of the angles of any triangle is equal to two right angles.*



Given $\triangle ABC$.

To Prove $\angle A + \angle B + \angle C =$ two rt. $\angle$s.

Proof. Produce AC to D, and draw line $CE \parallel AB$.

Then, $\angle ECD + \angle BCE + \angle ACB =$ two rt. $\angle$s. (1)

[The sum of all the $\angle$s on the same side of a str. line at a given point is equal to two rt. $\angle$s.] (§ 34)

Now since $\parallel$s AB and CE are cut by AD,

$$\angle ECD = \angle A.$$

[If two $\parallel$s are cut by a transversal, the corresp. $\angle$s are equal.] (§ 74)

And since $\parallel$s AB and CE are cut by BC,

$$\angle BCE = \angle B.$$

[If two $\parallel$s are cut by a transversal, the alt. int. $\angle$s are equal.] (§ 72)

Substituting in (1), we have

$$\angle A + \angle B + \angle ACB = \text{two rt. } \angle\text{s}.$$

85. Cor. I. It follows from the above demonstration that

$\angle BCD = \angle ECD + \angle BCE = \angle A + \angle B$; hence

1. *An exterior angle of a triangle is equal to the sum of the two opposite interior angles.*

2. *An exterior angle of a triangle is greater than either of the opposite interior angles.*

86. Cor. II. *If two triangles have two angles of one equal respectively to two angles of the other, the third angle of the first is equal to the third angle of the second.*

87. Cor. III. *A triangle cannot have two right angles, nor two obtuse angles.*

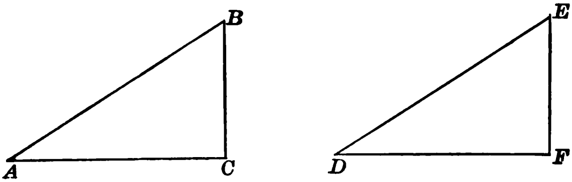
88. Cor. IV. *The sum of the acute angles of a right triangle is equal to one right angle.*

89. Cor. V. *Two right triangles are equal when a leg and an acute angle of one are equal respectively to a leg and the homologous acute angle of the other.*

The theorem follows by §§ 86 and 68.

PROP. XXVII. THEOREM.

90. *Two right triangles are equal when the hypotenuse and a leg of one are equal respectively to the hypotenuse and a leg of the other.*



Given, in rt. $\triangle$s ABC and DEF,

hypotenuse $AB =$ hypotenuse DE, and $BC = EF$.

To Prove $\triangle ABC = \triangle DEF$.

Proof. Superpose $\triangle ABC$ upon $\triangle DEF$ in such a way that side BC shall coincide with its equal EF; point B falling on point E, and point C on point F.

We have $\angle C = \angle F$.

[All rt. $\angle$s are equal.] (§ 26)

Then, side AC will fall on side DF.

But the equal oblique lines AB and DE cut off upon DF equal distances from the foot of $\perp EF$.

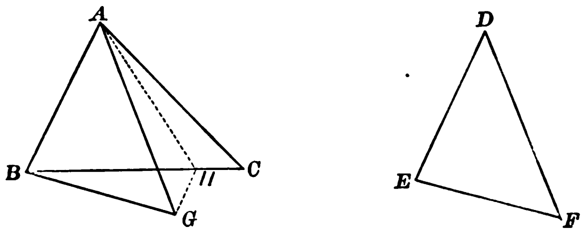
[If oblique lines be drawn from a point to a str. line, two equal oblique lines cut off equal distances from the foot of the $\perp$ from the point to the line.] (§ 50)

Therefore, point A falls on point D.

Hence, the $\triangle$s coincide throughout, and are equal.

PROP. XXVIII. THEOREM.

91. *If two triangles have two sides of one equal respectively to two sides of the other, but the included angle of the first greater than the included angle of the second, the third side of the first is greater than the third side of the second.*



Given, in $\triangle$ ABC and DEF,

$$AB = DE,\ AC = DF, \text{ and } \angle BAC > \angle D.$$

To Prove $BC > EF$.

Proof. Place $\triangle DEF$ in the position ABG; side DE coinciding with its equal AB, and vertex F falling at G.

Draw line AH bisecting $\angle GAC$, and meeting BC at H; also, draw line GH.

In $\triangle$ AGH and ACH, $AH = AH$.

Also, by hyp., $AG = AC$.

And by cons., $\angle GAH = \angle CAH$.

$$\therefore \triangle AGH = \triangle ACH.$$

[Two $\triangle$ are equal when two sides and the included $\angle$ of one are equal respectively to two sides and the included $\angle$ of the other.] (§ 63)

$$\therefore GH = CH.$$

[In equal figures, the homologous parts are equal.] (§ 66)

But, $BH + GH > BG$.

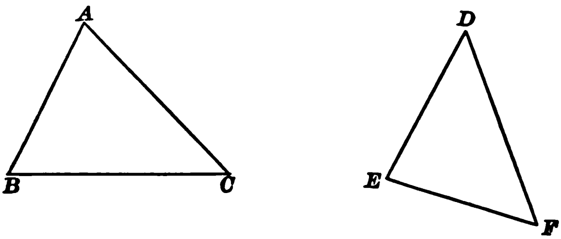
[A str. line is the shortest line between two points.] (Ax. 4)

Substituting for GH its equal CH, we have

$$BH + CH > BG, \text{ or } BC > EF.$$

Prop. XXIX. Theorem.

92. (Converse of Prop. XXVIII.) *If two triangles have two sides of one equal respectively to two sides of the other, but the third side of the first greater than the third side of the second, the included angle of the first is greater than the included angle of the second.*



Given, in ⊿ ABC and DEF,

$$AB = DE,\ AC = DF, \text{ and } BC > EF.$$

To Prove $\angle A > \angle D$.

Proof. We know that $\angle A$ is either $<$, equal to, or $>$ $\angle D$.

If we suppose $\angle A = \angle D$, $\triangle ABC$ would equal $\triangle DEF$.

[Two ⊿ are equal when two sides and the included $\angle$ of one are equal respectively to two sides and the included $\angle$ of the other.] (§ 63)

Then, BC would equal EF.

[In equal figures, the homologous parts are equal.] (§ 66)

Again, if we suppose $\angle A < \angle D$, BC would be $< EF$.

[If two ⊿ have two sides of one equal respectively to two sides of the other, but the included $\angle$ of the first $>$ the included $\angle$ of the second, the third side of the first is $>$ the third side of the second.] (§ 91)

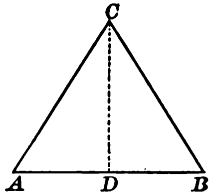
But each of these conclusions is contrary to the hypothesis that BC is $> EF$.

Then, if $\angle A$ can be neither equal to $\angle D$, nor $< \angle D$,

$$\angle A > \angle D.$$

PROP. XXX. THEOREM.

93. *In an isosceles triangle, the angles opposite the equal sides are equal.*



Given AC and BC the equal sides of isosceles $\triangle ABC$.

To Prove $\angle A = \angle B$.

Proof. Draw line $CD \perp AB$.

In rt. ⊿s ACD and BCD,

$$CD = CD.$$

And by hyp., $AC = BC$.

$$\therefore \triangle ACD = \triangle BCD.$$

[Two rt. ⊿s are equal when the hypotenuse and a leg of one are equal respectively to the hypotenuse and a leg of the other.] (§ 90)

$$\therefore \angle A = \angle B.$$

[In equal figures, the homologous parts are equal.] (§ 66)

94. Cor. I. From equal ⊿s ACD and BCD, we have

$$AD = BD, \text{ and } \angle ACD = \angle BCD; \text{ hence,}$$

1. *The perpendicular from the vertex to the base of an isosceles triangle bisects the base.*

2. *The perpendicular from the vertex to the base of an isosceles triangle bisects the vertical angle.*

95. Cor. II. *An equilateral triangle is also equiangular.*

PROP. XXXI. THEOREM.

96. (Converse of Prop. XXX.) *If two angles of a triangle are equal, the sides opposite are equal.*

(Fig. of Prop. XXX.)

Given, in $\triangle ABC$, $\angle A = \angle B$.

To Prove $AC = BC$.

(Prove $\triangle ACD = \triangle BCD$ by § 89.)

97. Cor. *An equiangular triangle is also equilateral.*

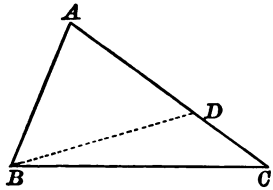
EXERCISES.

22. The angles A and B of a triangle ABC are 57° and 98° respectively; how many degrees are there in the exterior angle at C?

23. How many degrees are there in each angle of an equiangular triangle?

PROP. XXXII. THEOREM.

98. *If two sides of a triangle are unequal, the angles opposite are unequal, and the greater angle lies opposite the greater side.*



Given, in $\triangle ABC$, $AC > AB$.

To Prove $\angle ABC > \angle C$.

Proof. Take $AD = AB$, and draw line BD.

Then, in isosceles $\triangle ABD$,

$$\angle ABD = \angle ADB.$$

[In an isosceles $\triangle$, the $\angle$s opposite the equal sides are equal.] (§ 93)

Now since $\angle ADB$ is an ext. $\angle$ of $\triangle BDC$,

$$\angle ADB > \angle C.$$

[An ext. $\angle$ of a $\triangle$ is $>$ either of the opposite int. $\angle$s.] (§ 85)

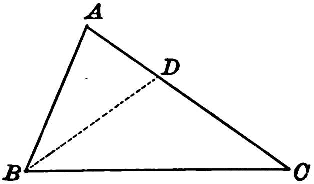
Therefore, its equal, $\angle ABD$, is $> \angle C$.

Then, since $\angle ABC$ is $> \angle ABD$, and $\angle ABD > \angle C$,

$$\angle ABC > \angle C.$$

PROP. XXXIII. THEOREM.

99. (Converse of Prop. XXXII.) *If two angles of a triangle are unequal, the sides opposite are unequal, and the greater side lies opposite the greater angle.*



Given, in $\triangle ABC$, $\angle ABC > \angle C$.

To Prove $AC > AB$.

Proof. Draw line BD, making $\angle CBD = \angle C$, and meeting AC at D.

Then, in $\triangle BCD$, $BD = CD$.

[If two $\angle$s of a $\triangle$ are equal, the sides opposite are equal.] (§ 96)

But, $AD + BD > AB$.

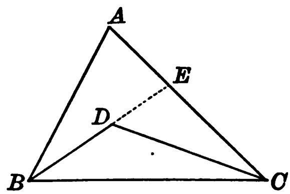
[A str. line is the shortest line between two points.] (Ax. 4)

Substituting for BD its equal CD, we have

$$AD + CD > AB, \text{ or } AC > AB.$$

PROP. XXXIV. THEOREM.

100. *If straight lines be drawn from a point within a triangle to the extremities of any side, the angle included by them is greater than the angle included by the other two sides.*



Given D, any point within $\triangle ABC$, and lines BD and CD.

To Prove $\angle BDC > \angle A$.

Proof. Produce BD to meet AC at E.

Then, since $\angle BDC$ is an ext. $\angle$ of $\triangle CDE$,

$$\angle BDC > \angle DEC.$$

[An ext. $\angle$ of a $\triangle$ is > either of the opposite int. $\angle$s.] (§ 85)

In like manner, since $\angle DEC$ is an ext. $\angle$ of $\triangle ABE$,

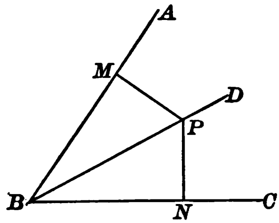
$$\angle DEC > \angle A.$$

Then, since $\angle BDC$ is > $\angle DEC$, and $\angle DEC > \angle A$,

$$\angle BDC > \angle A.$$

PROP. XXXV. THEOREM.

101. *Any point in the bisector of an angle is equally distant from the sides of the angle.*



Given P, any point in bisector BD of $\angle ABC$, and lines PM and $PN \perp$ to AB and BC, respectively.

To Prove $PM = PN$.

Proof. In rt. $\triangle$s BPM and BPN,

$$BP = BP.$$

And by hyp., $\angle PBM = \angle PBN$.

$$\therefore \triangle BPM = \triangle BPN.$$

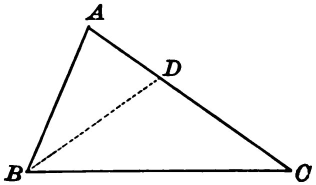
[Two rt. $\triangle$s are equal when the hypotenuse and an adj. $\angle$ of one are equal respectively to the hypotenuse and an adj. $\angle$ of the other.] (§ 70)

$$\therefore PM = PN.$$

[In equal figures, the homologous parts are equal.] (§ 66)

PROP. XXXIII. THEOREM.

99. (Converse of Prop. XXXII.) *If two angles of a triangle are unequal, the sides opposite are unequal, and the greater side lies opposite the greater angle.*



Given, in $\triangle ABC$, $\angle ABC > \angle C$.

To Prove $AC > AB$.

Proof. Draw line BD, making $\angle CBD = \angle C$, and meeting AC at D.

Then, in $\triangle BCD$, $BD = CD$.

[If two ∠s of a △ are equal, the sides opposite are equal.] (§ 96)

But, $AD + BD > AB$.

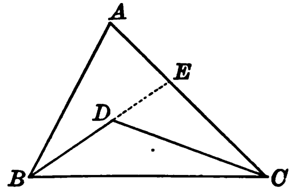
[A str. line is the shortest line between two points.] (Ax. 4)

Substituting for BD its equal CD, we have

$$AD + CD > AB, \text{ or } AC > AB.$$

PROP. XXXIV. THEOREM.

100. *If straight lines be drawn from a point within a triangle to the extremities of any side, the angle included by them is greater than the angle included by the other two sides.*



Given D, any point within $\triangle ABC$, and lines BD and CD.

To Prove $\angle BDC > \angle A$.

Proof. Produce BD to meet AC at E.

Then, since $\angle BDC$ is an ext. $\angle$ of $\triangle CDE$,

$$\angle BDC > \angle DEC.$$

[An ext. $\angle$ of a $\triangle$ is > either of the opposite int. $\angle$s.] (§ 85)

In like manner, since $\angle DEC$ is an ext. $\angle$ of $\triangle ABE$,

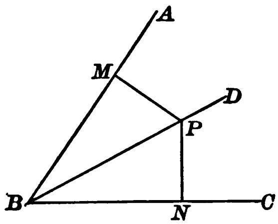
$$\angle DEC > \angle A.$$

Then, since $\angle BDC$ is > $\angle DEC$, and $\angle DEC > \angle A$,

$$\angle BDC > \angle A.$$

Prop. XXXV. Theorem.

101. *Any point in the bisector of an angle is equally distant from the sides of the angle.*



Given P, any point in bisector BD of $\angle ABC$, and lines PM and $PN \perp$ to AB and BC, respectively.

To Prove $PM = PN$.

Proof. In rt. $\triangle$s BPM and BPN,

$$BP = BP.$$

And by hyp., $\angle PBM = \angle PBN$.

$$\therefore \triangle BPM = \triangle BPN.$$

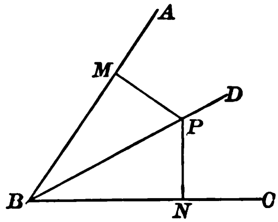
[Two rt. $\triangle$s are equal when the hypotenuse and an adj. $\angle$ of one are equal respectively to the hypotenuse and an adj. $\angle$ of the other.] (§ 70)

$$\therefore PM = PN.$$

[In equal figures, the homologous parts are equal.] (§ 66)

Prop. XXXVI. Theorem.

102. (Converse of Prop. XXXV.) *Every point which is within an angle, and equally distant from its sides, lies in the bisector of the angle.*



Given point P within $\angle ABC$, equally distant from sides AB and BC, and line BP.

To Prove $\angle PBM = \angle PBN$.

(Prove $\triangle BPM = \triangle BPN$, by § 90; the theorem then follows by § 66.)

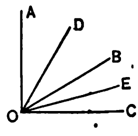
EXERCISES.

24. The angle at the vertex of an isosceles triangle ABC is equal to five-thirds the sum of the equal angles B and C. How many degrees are there in each angle?

25. If from a point O in a straight line AB lines OC and OD be drawn on opposite sides of AB, making $\angle AOC = \angle BOD$, prove that OC and OD lie in the same straight line.

(Fig. of Prop. IV. We have $\angle AOD + \angle BOD = 2$ rt. $\angle s$, and by hyp., $\angle BOD = \angle AOC$.)

26. If the bisectors of two adjacent angles make an angle of 45° with each other, the angles are complementary.



(Given OD and OE the bisectors of $\angle s\ AOB$ and BOC, respectively, and $\angle DOE = 45°$; to prove $\angle s\ AOB$ and BOC complementary.)

27. Prove Prop. XXX. by drawing CD to bisect $\angle ACB$. (§ 63.)

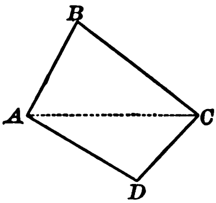
28. Prove Prop. XXX. by drawing CD to the middle point of AB.

29. Prove Prop. XXXI. by drawing CD to bisect $\angle ACB$. (§ 68.)

QUADRILATERALS.

DEFINITIONS.

103. A *quadrilateral* is a portion of a plane bounded by four straight lines; as *ABCD*.



The bounding lines are called the *sides* of the quadrilateral, and their points of intersection the *vertices*.

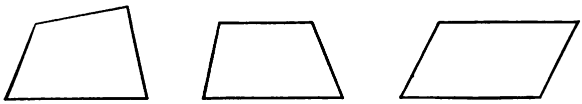
The *angles* of the quadrilateral are the angles included between the adjacent sides.

A *diagonal* is a straight line joining two opposite vertices; as *AC*.

104. A *Trapezium* is a quadrilateral no two of whose sides are parallel.

A *Trapezoid* is a quadrilateral two, and only two, of whose sides are parallel.

A *Parallelogram* (▱) is a quadrilateral whose opposite sides are parallel.



Trapezium. *Trapezoid.* *Parallelogram.*

The *bases* of a trapezoid are its parallel sides; the *altitude* is the perpendicular distance between them.

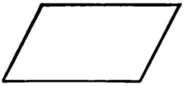
If either pair of parallel sides of a parallelogram be taken and called the *bases*, the *altitude* corresponding to these bases is the perpendicular distance between them.

105. A *Rhomboid* is a parallelogram whose angles are not right angles, and whose adjacent sides are unequal.

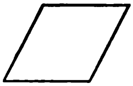
A *Rhombus* is a parallelogram whose angles are not right angles, and whose adjacent sides are equal.

A *Rectangle* is a parallelogram whose angles are right angles.

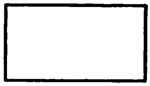
A *Square* is a rectangle whose sides are equal.



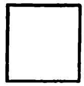
Rhomboid.



Rhombus.



Rectangle.



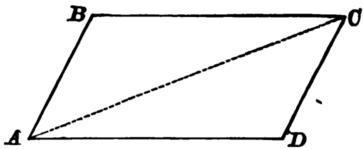
Square.

Prop. XXXVII. Theorem.

106. *In any parallelogram,*

I. *The opposite sides are equal.*

II. *The opposite angles are equal.*



Given ▱ $ABCD$.

I. **To Prove** $AB = CD$ and $BC = AD$.

Proof. Draw diagonal AC.

In $\triangle$s ABC and ACD, $AC = AC$.

Again, since ∥s BC and AD are cut by AC,

$$\angle BCA = \angle CAD.$$

[If two ∥s are cut by a transversal, the alt. int. $\angle$s are equal.] (§ 72)

In like manner, since ∥s AB and CD are cut by AC,

$$\angle BAC = \angle ACD.$$

$$\therefore \triangle ABC = \triangle ACD.$$

[Two $\triangle$s are equal when a side and two adj. $\angle$s of one are equal respectively to a side and two adj. $\angle$s of the other.] (§ 68)

$$\therefore AB = CD \text{ and } BC = AD.$$

[In equal figures, the homologous parts are equal.] (§ 66)

II. **To Prove** $\angle BAD = \angle BCD$ and $\angle B = \angle D$.

Proof. We have $AB \parallel CD$, and $AD \parallel CB$; and AB and CD, and also AD and CB, extend in opposite directions from A and C.

$\therefore \angle BAD = \angle BCD.$

[Two ∠s whose sides are ∥, each to each, are equal if both pairs of ∥ sides extend in opposite directions from their vertices.] (§ 81)

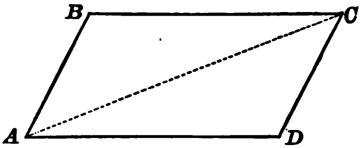
In like manner, $\angle B = \angle D.$

107. Cor. I. *Parallel lines included between parallel lines are equal.*

108. Cor. II. *A diagonal of a parallelogram divides it into two equal triangles.*

Prop. XXXVIII. Theorem.

109. (Converse of Prop. XXXVII, I.) *If the opposite sides of a quadrilateral are equal, the figure is a parallelogram.*



Given, in quadrilateral $ABCD$,

$AB = CD$ and $BC = AD.$

To Prove $ABCD$ a ▱.

Proof. Draw diagonal AC.

In △s ABC and ACD, $AC = AC.$

And by hyp., $AB = CD$ and $BC = AD.$

$\therefore \triangle ABC = \triangle ACD.$

[Two △s are equal when the three sides of one are equal respectively to the three sides of the other.] (§ 69)

$\therefore \angle BCA = \angle CAD$ and $\angle BAC = \angle ACD.$

[In equal figures, the homologous parts are equal.] (§ 66)

Since $\angle BCA = \angle CAD$, $BC \parallel AD.$

[If two str. lines are cut by a transversal, and the alt. int. ∠s are equal, the two lines are ∥.] (§ 73)

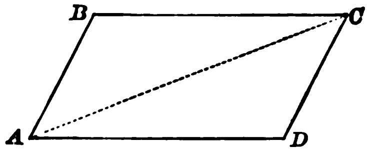
In like manner, since $\angle BAC = \angle ACD$, $AB \parallel CD.$

Then by def., $ABCD$ is a ▱.

Ex. 30. If one angle of a parallelogram is 119°, how many degrees are there in each of the others?

PROP. XXXIX. THEOREM.

110. *If two sides of a quadrilateral are equal and parallel, the figure is a parallelogram.*



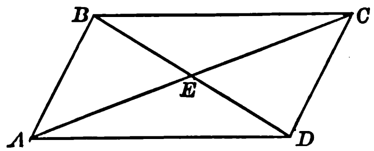
Given, in quadrilateral $ABCD$, BC equal and ∥ to AD.

To Prove $ABCD$ a ▱.

(Prove $\triangle ABC = \triangle ACD$, by § 63; then, the other two sides of the quadrilateral are equal, and the theorem follows by § 109.)

PROP. XL. THEOREM.

111. *The diagonals of a parallelogram bisect each other.*



Given diagonals AC and BD of ▱ $ABCD$ intersecting at E.

To Prove $AE = EC$ and $BE = ED$.

(Prove $\triangle AED = \triangle BEC$, by § 68.)

Note. The point E is called the *centre* of the parallelogram.

PROP. XLI. THEOREM.

112. (Converse of Prop. XL.) *If the diagonals of a quadrilateral bisect each other, the figure is a parallelogram.*

(Fig. of Prop. XL.)

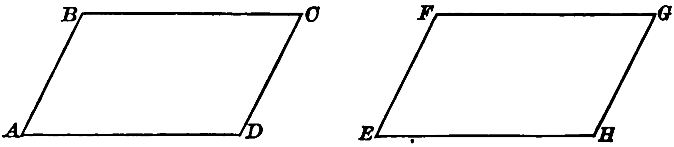
Given *AC* and *BD*, the diagonals of quadrilateral *ABCD*, bisecting each other at *E*.

To Prove *ABCD* a ▱.

(Prove △ *AED* = △ *BEC*, by § 63; then *AD* = *BC*; in like manner, *AB* = *CD*, and the theorem follows by § 109.)

PROP. XLII. THEOREM.

113. *Two parallelograms are equal when two adjacent sides and the included angle of one are equal respectively to two adjacent sides and the included angle of the other.*



Given, in ▱s *ABCD* and *EFGH*,

AB = *EF*, *AD* = *EH*, and ∠ *A* = ∠ *E*.

To Prove ▱ *ABCD* = ▱ *EFGH*.

Proof. Superpose ▱ *ABCD* upon ▱ *EFGH* in such a way that ∠ *A* shall coincide with its equal ∠ *E*; side *AB* falling on side *EF*, and side *AD* on side *EH*.

Then since *AB* = *EF* and *AD* = *EH*, point *B* will fall on point *F*, and point *D* on point *H*.

Now since *BC* ∥ *AD* and *FG* ∥ *EH*, side *BC* will fall on side *FG*, and point *C* will fall somewhere on *FG*.

[But one str. line can be drawn through a given point ∥ to a given str. line.] (§ 53)

In like manner, side *DC* will fall on side *HG*, and point *C* will fall somewhere on *HG*.

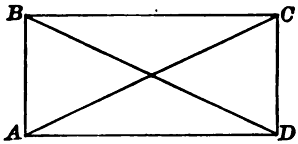
Then point *C*, falling at the same time on *FG* and *HG*, must fall at their intersection *G*.

Hence, the ▱s coincide throughout, and are equal.

114. Cor. *Two rectangles are equal if the base and altitude of one are equal respectively to the base and altitude of the other.*

PROP. XLIII. THEOREM.

115. *The diagonals of a rectangle are equal.*



Given AC and BD the diagonals of rect. $ABCD$.

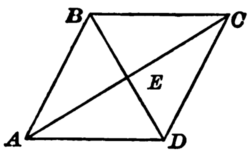
To Prove $AC = BD$.

(Prove rt. $\triangle ABD =$ rt. $\triangle ACD$, by § 63.)

116. Cor. *The diagonals of a square are equal.*

PROP. XLIV. THEOREM.

117. *The diagonals of a rhombus bisect each other at right angles.*



(AC and BD bisect each other at rt. ∠s by § 43.)

EXERCISES.

31. The bisector of the vertical angle of an isosceles triangle bisects the base at right angles.

(Fig. of Prop. XXX. In equal △s ACD and BCD, we have $\angle ADC = \angle BDC$; then $CD \perp AB$ by § 24.)

32. The line joining the vertex of an isosceles triangle to the middle point of the base, is perpendicular to the base, and bisects the vertical angle.

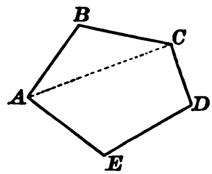
(Fig. of Prop. XXX. Prove $CD \perp AB$ as in Ex. 31.)

33. If one angle of a parallelogram is a right angle, the figure is a rectangle.

POLYGONS.

DEFINITIONS.

118. A *polygon* is a portion of a plane bounded by three or more straight lines; as *ABCDE*.



The bounding lines are called the *sides* of the polygon, and their sum is called the *perimeter*.

The *angles* of the polygon are the angles *EAB, ABC,* etc., included between the adjacent sides; and their vertices are called the *vertices* of the polygon.

A *diagonal* of a polygon is a straight line joining any two vertices which are not consecutive; as *AC*.

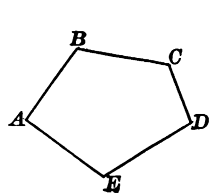
119. Polygons are classified with reference to the number of their sides, as follows:

No. of Sides.	Designation.	No. of Sides.	Designation.
3	Triangle.	8	Octagon.
4	Quadrilateral.	9	Enneagon.
5	Pentagon.	10	Decagon.
6	Hexagon.	11	Hendecagon.
7	Heptagon.	12	Dodecagon.

120. An *equilateral* polygon is a polygon all of whose sides are equal.

An *equiangular* polygon is a polygon all of whose angles are equal.

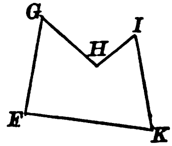
121. A polygon is called *convex* when no side, if produced, will enter the surface enclosed by the perimeter; as *ABCDE*.



It is evident that, in such a case, each angle of the polygon is less than two right angles.

All polygons considered hereafter will be understood to be convex, unless the contrary is stated.

A polygon is called *concave* when at least two of its sides, if produced, will enter the surface enclosed by the perimeter; as $FGHIK$.

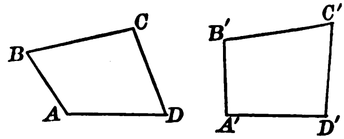


It is evident that, in such a case, at least one angle of the polygon is greater than two right angles.

Thus, in polygon $FGHIK$, the interior angle GHI is greater than two right angles.

Such an angle is called *re-entrant.*

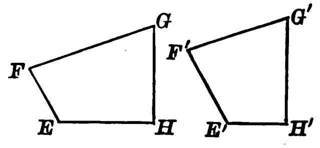
122. Two polygons are said to be ***mutually equilateral*** when the sides of one are equal respectively to the sides of the other, when taken in the same order.



Thus, polygons $ABCD$ and $A'B'C'D'$ are mutually equilateral if

$$AB = A'B',\ BC = B'C',\ CD = C'D',\ \text{and}\ DA = D'A'.$$

Two polygons are said to be ***mutually equiangular*** when the angles of one are equal respectively to the angles of the other when taken in the same order.



Thus, polygons $EFGH$ and $E'F'G'H'$ are mutually equiangular if

$$\angle E = \angle E',\ \angle F = \angle F',\ \angle G = \angle G',\ \text{and}\ \angle H = \angle H'.$$

123. In polygons which are mutually equilateral or mutually equiangular, sides or angles which are similarly placed are called *homologous.*

In mutually equiangular polygons, the sides included between equal angles are homologous.

124. If two *triangles* are mutually equilateral, they are also mutually equiangular (§ 69).

But with this exception, two polygons may be mutually equilateral without being mutually equiangular, or mutually equiangular without being mutually equilateral.

If two polygons are both mutually equilateral and mutually equiangular, they are equal.

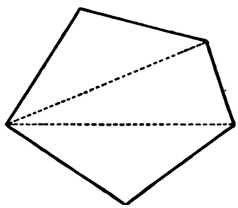
For they can evidently be applied one to the other so as to coincide throughout.

125. *Two polygons are equal when they are composed of the same number of triangles, equal each to each, and similarly placed.*

For they can evidently be applied one to the other so as to coincide throughout.

PROP. XLV. THEOREM.

126. *The sum of the angles of any polygon is equal to two right angles taken as many times, less two, as the polygon has sides.*



Given a polygon of n sides.

To Prove the sum of its ∠s equal to $n-2$ times two rt. ∠s.

Proof. The polygon may be divided into $n-2$ △s by drawing diagonals from one of its vertices.

The sum of the ∠s of the polygon is equal to the sum of the ∠s of the △s.

But the sum of the ∠s of each △ is two rt. ∠s.

[The sum of the ∠s of any △ is equal to two rt. ∠s.] (§ 84)

Hence, the sum of the ∠s of the polygon is $n-2$ times two rt. ∠s.

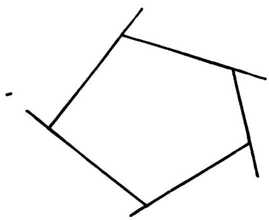
127. Cor. I. *The sum of the angles of any polygon is equal to twice as many right angles as the polygon has sides, less four right angles.*

For if R represents a rt. ∠, and n the number of sides of a polygon, the sum of its ∠s is $(n-2) \times 2R$, or $2nR - 4R$.

128. Cor. II. *The sum of the angles of a quadrilateral is equal to four right angles; of a pentagon, six right angles; of a hexagon, eight right angles; etc.*

Prop. XLVI. Theorem.

129. *If the sides of any polygon be produced so as to make an exterior angle at each vertex, the sum of these exterior angles is equal to four right angles.*



Given a polygon of n sides with its sides produced so as to make an ext. ∠ at each vertex.

To Prove the sum of these ext. ∠s equal to 4 rt. ∠s.

Proof. The sum of the ext. and int. ∠s at any one vertex is two rt. ∠s.

[If two adj. ∠s have their ext. sides in the same str. line, their sum is equal to two rt. ∠s.] (§ 32)

Hence, the sum of *all* the ext. and int. ∠s is $2n$ rt. ∠s.
But the sum of the int. ∠s alone is $2n$ rt. ∠s − 4 rt. ∠s.

[The sum of the ∠s of any polygon is equal to twice as many rt. ∠s as the polygon has sides, less 4 rt. ∠s.] (§ 127)

Whence, the sum of the ext. ∠s is 4 rt. ∠s.

EXERCISES.

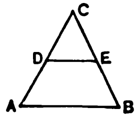
34. How many degrees are there in each angle of an equiangular hexagon? of an equiangular octagon? of an equiangular decagon? of an equiangular dodecagon?

35. How many degrees are there in the exterior angle at each vertex of an equiangular pentagon?

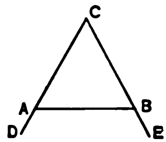
36. If two angles of a quadrilateral are supplementary, the other two angles are supplementary.

37. If, in a triangle ABC, $\angle A = \angle B$, a line parallel to AB makes equal angles with sides AC and BC.

(To prove $\angle CDE = \angle CED$.)



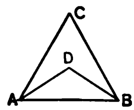
38. If the equal sides of an isosceles triangle be produced, the exterior angles made with the base are equal. (§ 31, 2.)



39. If the perpendicular from the vertex to the base of a triangle bisects the base, the triangle is isosceles.

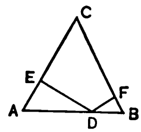
(Fig. of Prop. XXX. ⊿s ACD and BCD are equal by § 63.)

40. The bisectors of the equal angles of an isosceles triangle form, with the base, another isosceles triangle.



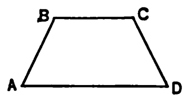
41. If from any point in the base of an isosceles triangle perpendiculars to the equal sides be drawn, they make equal angles with the base.

($\angle ADE = \angle BDF$, by § 31, 1.)



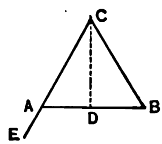
42. If the angles adjacent to one base of a trapezoid are equal, those adjacent to the other base are also equal.

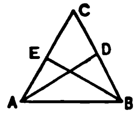
(Given $\angle A = \angle D$; to prove $\angle B = \angle C$.)



43. Either exterior angle at the base of an isosceles triangle is equal to the sum of a right angle and one-half the vertical angle.

($\angle DAE$ is an ext. $\angle$ of $\triangle ACD$.)



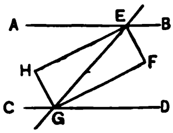


44. The straight lines bisecting the equal angles of an isosceles triangle, and terminating in the opposite sides, are equal.

($\triangle ABD = \triangle ABE$.)

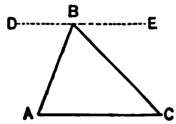
45. Two isosceles triangles are equal when the base and vertical angle of one are equal respectively to the base and vertical angle of the other.

(Each of the remaining ∠s of one △ is equal to each of the remaining ∠s of the other.)



46. If two parallels are cut by a transversal, the bisectors of the four interior angles form a rectangle.

($EH \parallel FG$, by § 73; in like manner, $EF \parallel GH$; then use Exs. 12 and 33.)



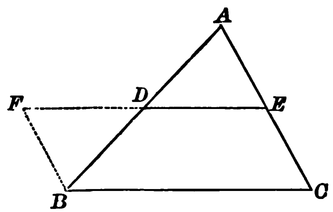
47. Prove Prop. XXVI. by drawing through B a line parallel to AC.

(Sum of ∠s at B = 2 rt. ∠s.)

MISCELLANEOUS THEOREMS.

PROP. XLVII. THEOREM.

130. *The line joining the middle points of two sides of a triangle is parallel to the third side, and equal to one-half of it.*



Given line DE joining middle points of sides AB and AC, respectively, of $\triangle ABC$.

To Prove $DE \parallel BC$, and $DE = \frac{1}{2} BC$.

Proof. Draw line $BF \parallel AC$, meeting ED produced at F.

In $\triangle$s ADE and BDF,

$$\angle ADE = \angle BDF.$$

[If two str. lines intersect, the vertical $\angle$s are equal.] (§ 40)

Also, since ||s AC and BF are cut by AB,

$$\angle A = \angle DBF.$$

[If two ||s are cut by a transversal, the alt. int. $\angle$s are equal.] (§ 72)

And by hyp., $AD = BD$.

$$\therefore \triangle ADE = \triangle BDF.$$

[Two $\triangle$s are equal when a side and two adj. $\angle$s of one are equal respectively to a side and two adj. $\angle$s of the other.] (§ 68)

$$\therefore DE = DF \text{ and } AE = BF.$$

[In equal figures, the homologous parts are equal.] (§ 66)

Then since, by hyp., $AE = EC$, BF is equal and || to CE.

Whence, $BCEF$ is a ▱.

[If two sides of a quadrilateral are equal and ||, the figure is a ▱.] (§ 110)

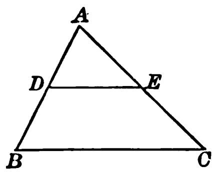
$$\therefore DE \parallel BC.$$

Again, since $DE = DF$,

$$DE = \tfrac{1}{2} FE = \tfrac{1}{2} BC.$$

[In any ▱, the opposite sides are equal.] (§ 106)

131. Cor. *The line which bisects one side of a triangle, and is parallel to another side, bisects also the third side.*



Given, in $\triangle ABC$, D the middle point of side AB, and line $DE \parallel BC$.

To Prove that DE bisects AC.

Proof. A line joining D to the middle point of AC will be $\parallel BC$.

[The line joining the middle points of two sides of a $\triangle$ is || to the third side.] (§ 130)

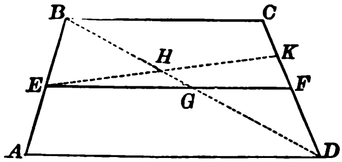
Then this line will coincide with DE.

[But one str. line can be drawn through a given point || to a given str. line.] (§ 53)

Therefore, DE bisects AC.

PROP. XLVIII. THEOREM.

132. *The line joining the middle points of the non-parallel sides of a trapezoid is parallel to the bases, and equal to one-half their sum.*



Given line EF joining middle points of non-∥ sides AB and CD, respectively, of trapezoid $ABCD$.

To Prove EF ∥ to AD and BC, and $EF = \frac{1}{2}(AD + BC)$.

Proof. If EF is not ∥ to AD and BC, draw line EK ∥ to AD and BC, meeting CD at K; and draw line BD intersecting EF at G, and EK at H.

In $\triangle ABD$, EH is ∥ AD and bisects AB; then it bisects BD.

[The line which bisects one side of a △, and is ∥ to another side, bisects also the third side.] (§ 131)

In like manner, in $\triangle BCD$, HK is ∥ BC and bisects BD; then it bisects CD.

But this is impossible unless EK coincides with EF.

[But one str. line can be drawn between two points.] (Ax. 3)

Hence, EF is ∥ to AD and BC.

Again, since EG coincides with EH, and EH bisects AB and BD,

$$EG = \frac{1}{2} AD. \quad (1)$$

[The line joining the middle points of two sides of a △ is equal to one-half the third side.] (§ 130)

In like manner, since GF bisects BD and CD,

$$GF = \frac{1}{2} BC. \quad (2)$$

Adding (1) and (2),

$$EG + GF = \frac{1}{2} AD + \frac{1}{2} BC.$$

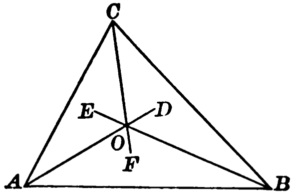
Or,

$$EF = \frac{1}{2}(AD + BC).$$

133. Cor. *The line which is parallel to the bases of a trapezoid, and bisects one of the non-parallel sides, bisects the other also.*

Prop. XLIX. Theorem.

134. *The bisectors of the angles of a triangle intersect at a common point.*



Given lines AD, BE, and CF bisecting $\angle$s A, B, and C, respectively, of $\triangle ABC$.

To Prove that AD, BE, and CF intersect at a common point.

Proof. Let AD and BE intersect at O.

Since O is in bisector AD, it is equally distant from sides AB and AC.

[Any point in the bisector of an $\angle$ is equally distant from the sides of the $\angle$.] (§ 101)

In like manner, since O is in bisector BE, it is equally distant from sides AB and BC.

Then O is equally distant from sides AC and BC, and therefore lies in bisector CF.

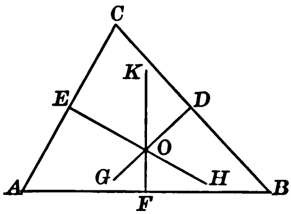
[Every point which is within an $\angle$, and equally distant from its sides, lies in the bisector of the $\angle$.] (§ 102)

Hence, AD, BE, and CF intersect at the common point O.

135. Cor. *The point of intersection of the bisectors of the angles of a triangle is equally distant from the sides of the triangle.*

PROP. L. THEOREM.

136. *The perpendiculars erected at the middle points of the sides of a triangle intersect at a common point.*



Given DG, EH, and FK the ⊥s erected at middle points D, E, and F, of sides BC, CA, and AB, respectively, of $\triangle ABC$.

To Prove that DG, EH, and FK intersect at a common point.

(Let DG and EH intersect at O; by § 41, O is equally distant from B and C; it is also equally distant from A and C; the theorem follows by § 42.)

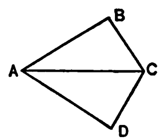
137. Cor. *The point of intersection of the perpendiculars erected at the middle points of the sides of a triangle, is equally distant from the vertices of the triangle.*

EXERCISES.

48. If the diagonals of a parallelogram are equal, the figure is a rectangle.

(Fig. of Prop. XLIII. ⊿s ABD and ACD are equal, and therefore $\angle BAD = \angle ADC$; also, these ∠s are supplementary.)

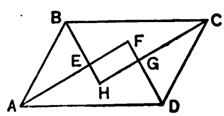
49. If two adjacent sides of a quadrilateral are equal, and the diagonal bisects their included angle, the other two sides are equal.



(Given $AB = AD$, and AC bisecting $\angle BAD$; to prove $BC = CD$.)

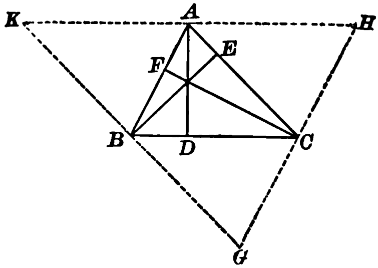
50. The bisectors of the interior angles of a parallelogram form a rectangle.

(By Ex. 46, each ∠ of $EFGH$ is a rt. ∠.)



PROP. LI. THEOREM.

138. *The perpendiculars from the vertices of a triangle to the opposite sides intersect at a common point.*



Given *AD, BE,* and *CF* the ⊥s from the vertices of △ *ABC* to the opposite sides.

To Prove that *AD, BE,* and *CF* intersect at a common point.

Proof. Through *A, B,* and *C,* draw lines *HK, KG,* and *GH* ∥ to *BC, CA,* and *AB,* respectively, forming △ *GHK.*

Then *AD,* being ⊥ *BC,* is also ⊥ *HK.*

[A str. line ⊥ to one of two ∥s is ⊥ to the other.] (§ 56)

Now since, by cons., *ABCH* and *ACBK* are ▱s,

$$AH = BC \text{ and } AK = BC.$$

[In any ▱, the opposite sides are equal.] (§ 106)

$$\therefore AH = AK.$$

[Things which are equal to the same thing, are equal to each other.] (Ax. 1)

Then *AD* is ⊥ *HK* at the middle point of *HK.*

In like manner, *BE* and *CF* are ⊥ to *KG* and *GH,* respectively, at their middle points.

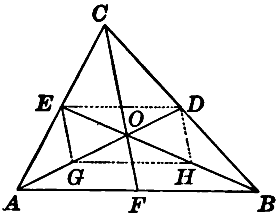
Then, *AD, BE,* and *CF* being ⊥ to the sides of △ *GHK* at their middle points, intersect at a common point.

[The ⊥ erected at the middle points of the sides of a △ intersect at a common point.] (§ 136)

139. Def. A *median* of a triangle is a line drawn from any vertex to the middle point of the opposite side.

PROP. LII. THEOREM.

140. *The medians of a triangle intersect at a common point, which lies two-thirds the way from each vertex to the middle point of the opposite side.*



Given AD, BE, and CF the medians of $\triangle ABC$.

To Prove that AD, BE, and CF intersect at a common point, which lies two-thirds the way from each vertex to the middle point of the opposite side.

Proof. Let AD and BE intersect at O.

Let G and H be the middle points of OA and OB, respectively, and draw lines ED, GH, EG, and DH.

Since ED bisects AC and BC,

$$ED \parallel AB \text{ and } = \tfrac{1}{2} AB.$$

[The line joining the middle points of two sides of a △ is ∥ to the third side, and equal to one-half of it.] (§ 130)

In like manner, since GH bisects OA and OB,

$$GH \parallel AB \text{ and } = \tfrac{1}{2} AB.$$

Then ED and GH are equal and ∥.

[Things which are equal to the same thing, are equal to each other.] (Ax. 1)

[Two str. lines ∥ to the same str. line are ∥ to each other.] (§ 55)

Therefore, $EDHG$ is a ▱.

[If two sides of a quadrilateral are equal and ∥, the figure is a ▱.] (§ 110)

Then GD and EH bisect each other at O.

[The diagonals of a ▱ bisect each other.] (§ 111)

But by hyp., G is the middle point of OA, and H of OB.

$$\therefore AG = OG = OD, \text{ and } BH = OH = OE.$$

That is, AD and BE intersect at a point O which lies two-thirds the way from A to D, and from B to E.

In like manner, AD and CF intersect at a point which lies two-thirds the way from A to D, and from C to F.

Hence, AD, BE, and CF intersect at the common point O, which lies two-thirds the way from each vertex to the middle point of the opposite side.

LOCI.

141. Def. If a series of points, all of which satisfy a certain condition, lie in a certain line, and every point in this line satisfies the given condition, the line is said to be the *locus* of the points.

For example, every point which satisfies the condition of being equally distant from the extremities of a straight line, lies in the perpendicular erected at the middle point of the line (§ 42).

Also, every point in the perpendicular erected at the middle point of a line satisfies the condition of being equally distant from the extremities of the line (§ 41).

Hence, *the perpendicular erected at the middle point of a straight line is the* **Locus** *of points which are equally distant from the extremities of the line.*

Again, every point which satisfies the condition of being within an angle, and equally distant from its sides, lies in the bisector of the angle (§ 102).

Also, every point in the bisector of an angle satisfies the condition of being equally distant from its sides (§ 101).

Hence, *the bisector of an angle is the locus of points which are within the angle, and equally distant from its sides.*

EXERCISES.

51. Two straight lines are parallel if any two points of either are equally distant from the other.

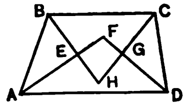
(Prove by *Reductio ad Absurdum*.)

52. What is the locus of points at a given distance from a given straight line? (Ex. 51.)

53. What is the locus of points equally distant from a pair of intersecting straight lines?

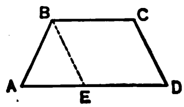
54. What is the locus of points equally distant from a pair of parallel straight lines?

55. The bisectors of the interior angles of a trapezoid form a quadrilateral, two of whose angles are right angles. (Ex. 46.)



56. If the angles at the base of a trapezoid are equal, the non-parallel sides are also equal.

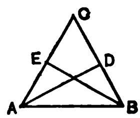
(Given $\angle A = \angle D$; to prove $AB = CD$. Draw $BE \parallel CD$.)



57. If the non-parallel sides of a trapezoid are equal, the angles which they make with the bases are equal.

(Fig. of Ex. 56. Given $AB = CD$; to prove $\angle A = \angle D$, and also $\angle ABC = \angle C$. Draw $BE \parallel CD$.)

58. The perpendiculars from the extremities of the base of an isosceles triangle to the opposite sides are equal.

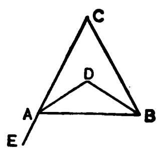


59. If the perpendiculars from the extremities of the base of a triangle to the opposite sides are equal, the triangle is isosceles.

(Converse of Ex. 58. Prove $\triangle ACD = \triangle BCE$.)

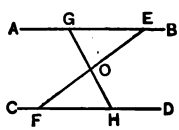
60. The angle between the bisectors of the equal angles of an isosceles triangle is equal to the exterior angle at the base of the triangle.

($\angle ADB = 180° - (\angle BAD + \angle ABD)$.)

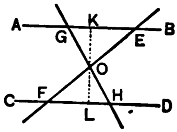


61. If a line joining two parallels be bisected, any line drawn through the point of bisection and included between the parallels will be bisected at the point.

(To prove that GH is bisected at O.)

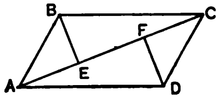


62. If through a point midway between two parallels two transversals be drawn, they intercept equal portions of the parallels.

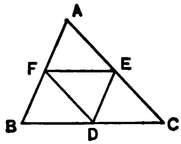


(Draw $OK \perp AB$, and produce KO to meet CD at L. Then $\triangle OGK = \triangle OHL$.)

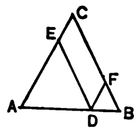
63. If perpendiculars BE and DF be drawn from vertices B and D of parallelogram $ABCD$ to the diagonal AC, prove $BE = DF$. (§ 70.)



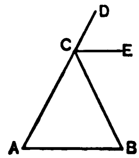
64. The lines joining the middle points of the sides of a triangle divide it into four equal triangles. (§ 130.)



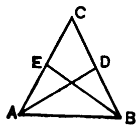
65. If from any point in the base of an isosceles triangle parallels to the equal sides be drawn, the perimeter of the parallelogram formed is equal to the sum of the equal sides of the triangle. (§ 96.)



66. The bisector of the exterior angle at the vertex of an isosceles triangle is parallel to the base. (§ 85, 1.)

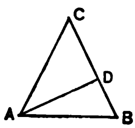


67. The medians drawn from the extremities of the base of an isosceles triangle are equal.



68. If from the vertex of one of the equal angles of an isosceles triangle a perpendicular be drawn to the opposite side, it makes with the base an angle equal to one-half the vertical angle of the triangle.

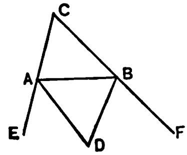
(To prove $\angle BAD = \frac{1}{2} \angle C$.)



69. If the exterior angles at the vertices A and B of triangle ABC are bisected by lines which meet at D, prove

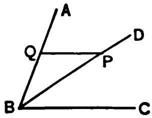
$$\angle ADB = 90° - \tfrac{1}{2} C.$$

$(\angle ADB = 180° - (\angle BAD + \angle ABD).)$

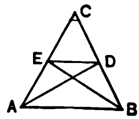


70. The diagonals of a rhombus bisect its angles.
(Fig. of Prop. XLIV.)

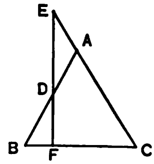
71. If from any point in the bisector of an angle a parallel to one of the sides be drawn, the bisector, the parallel, and the remaining side form an isosceles triangle.



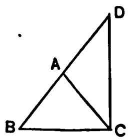
72. If the bisectors of the equal angles of an isosceles triangle meet the equal sides at D and E, prove DE parallel to the base of the triangle.
(Prove $\triangle\ CED$ isosceles.)



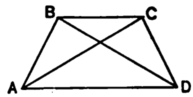
73. If at any point D in one of the equal sides AB of isosceles triangle ABC, DE be drawn perpendicular to base BC meeting CA produced at E, prove triangle ADE isosceles.



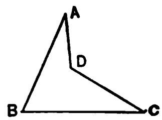
74. From C, one of the extremities of the base BC of isosceles triangle ABC, a line is drawn meeting BA produced at D, making $AD = AB$. Prove CD perpendicular to BC. (§ 84.)
($\triangle\ ACD$ is isosceles.)



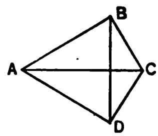
75. If the non-parallel sides of a trapezoid are equal, its diagonals are also equal. (Ex. 57.)



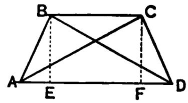
76. If ADC is a re-entrant angle of quadrilateral $ABCD$, prove that angle ADC, exterior to the figure, is equal to the sum of interior angles A, B, and C. (§ 128.)



77. If a diagonal of a quadrilateral bisects two of its angles, it is perpendicular to the other diagonal.
(Prove $AC \perp DB$, by § 43.)

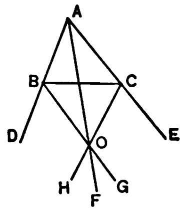


78. In a quadrilateral $ABCD$, angles ABD and CAD are equal to ACD and BDA, respectively; prove BC parallel to AD.
(Prove $AB = CD$; then prove $BE = CF$.)



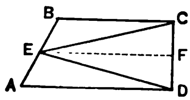
79. State and prove the converse of Prop. XLIV. (§ 41, I.)

80. State and prove the converse of Ex. 66, p. 67. (§ 96.)



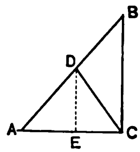
81. The bisectors of the exterior angles at two vertices of a triangle, and the bisector of the interior angle at the third vertex meet at a common point.

(Prove as in § 134.)



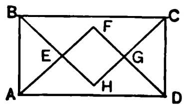
82. *ABCD* is a trapezoid whose parallel sides *AD* and *BC* are perpendicular to *CD*. If *E* is the middle point of *AB*, prove $EC = ED$. (§ 41, I.)

(Draw $EF \parallel AD$.)



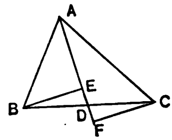
83. The middle point of the hypotenuse of a right triangle is equally distant from the vertices of the triangle.

(To prove $AD = BD = CD$. Draw $DE \parallel BC$.)

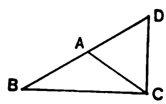


84. The bisectors of the angles of a rectangle form a square.

(By Ex. 50, *EFGH* is a rectangle. Now prove $AF = BH$ and $AE = BE$.)



85. If *D* is the middle point of side *BC* of triangle *ABC*, and *BE* and *CF* are perpendiculars from *B* and *C* to *AD*, produced if necessary, prove $BE = CF$.



86. The angle at the vertex of isosceles triangle *ABC* is equal to twice the sum of the equal angles *B* and *C*. If *CD* be drawn perpendicular to *BC*, meeting *BA* produced at *D*, prove triangle *ACD* equilateral.

(Prove each $\angle$ of $\triangle ACD$ equal to 60°.)

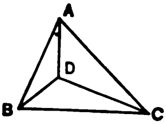
87. If angle *B* of triangle *ABC* is greater than angle *C*, and *BD* be drawn to *AC* making $AD = AB$, prove

$$\angle ADB = \tfrac{1}{2}(B + C), \text{ and } \angle CBD = \tfrac{1}{2}(B - C).$$

(Fig. of Prop. XXXII.)

88. How many sides are there in the polygon the sum of whose interior angles exceeds the sum of its exterior angles by 540°?

89. The sum of the lines drawn from any point within a triangle to the vertices is greater than the half-sum of the three sides.

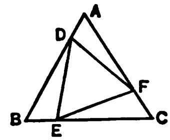


(Apply § 61 to each of the ▲ *ABD*, *ACD*, and *BCD*.)

90. The sum of the lines drawn from any point within a triangle to the vertices is less than the sum of the three sides. (§ 48.)

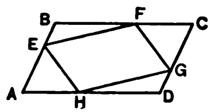
(Fig. of Ex. 89.)

91. If *D*, *E*, and *F* are points on the sides *AB*, *BC*, and *CA*, respectively, of equilateral triangle *ABC*, such that *AD* = *BE* = *CF*, prove *DEF* an equilateral triangle.

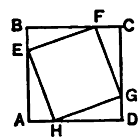


(Prove ▲ *ADF*, *BDE*, and *CEF* equal.)

92. If *E*, *F*, *G*, and *H* are points on the sides *AB*, *BC*, *CD*, and *DA*, respectively, of parallelogram *ABCD*, such that *AE* = *CG* and *BF* = *DH*, prove *EFGH* a parallelogram.

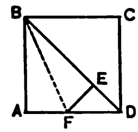


93. If *E*, *F*, *G*, and *H* are points on sides *AB*, *BC*, *CD*, and *DA*, respectively, of square *ABCD*, such that *AE* = *BF* = *CG* = *DH*, prove *EFGH* a square.

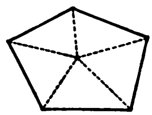


(First prove *EFGH* equilateral. Then prove ∠ *FEH* = 90°.)

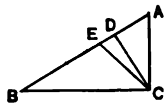
94. If on the diagonal *BD* of square *ABCD* a distance *BE* be taken equal to *AB*, and *EF* be drawn perpendicular to *BD*, meeting *AD* at *F*, prove that *AF* = *EF* = *ED*.



95. Prove the theorem of § 127 by drawing lines from any point within the polygon to the vertices. (§ 35.)



96. If *CD* is the perpendicular from the vertex of the right angle to the hypotenuse of right triangle *ABC*, and *CE* the bisector of angle *C*, meeting *AB* at *E*, prove ∠ *DCE* equal to one-half the difference of angles *A* and *B*.



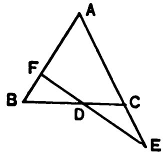
(To prove $\angle DCE = \frac{1}{2}(\angle A - \angle B)$.)

97. State and prove the converse of Ex. 70, p. 68.

(Fig. of Prop. XLIV. Prove the sides all equal.)

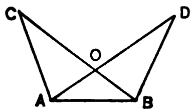
98. State and prove the converse of Ex. 75, p. 68.

(Fig. of Ex. 78. Prove $\triangle\triangle$ ACF and BDE equal.)

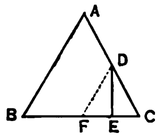


99. D is any point in base BC of isosceles triangle ABC. The side AC is produced from C to E, so that $CE = CD$, and DE is drawn meeting AB at F. Prove $\angle AFE = 3\angle AEF$.

($\angle AFE$ is an ext. $\angle$ of $\triangle BFD$.)

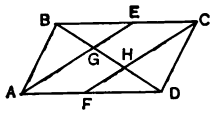


100. If ABC and ABD are two triangles on the same base and on the same side of it, such that $AC = BD$ and $AD = BC$, and AD and BC intersect at O, prove triangle OAB isosceles.



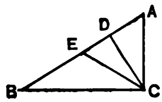
101. If D is the middle point of side AC of equilateral triangle ABC, and DE be drawn perpendicular to BC, prove $EC = \frac{1}{4}BC$.

(Draw DF to the middle point of BC.)



102. If in parallelogram $ABCD$, E and F are the middle points of sides BC and AD, respectively, prove that lines AE and CF trisect diagonal BD.

(By § 131, AE bisects BH, and CF bisects DG.)



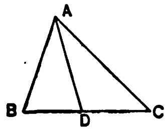
103. If CD is the perpendicular from C to the hypotenuse of right triangle ABC, and E is the middle point of AB, prove $\angle DCE$ equal to the difference of angles A and B. (Ex. 83.)

104. If one acute angle of a right triangle is double the other, the hypotenuse is double the shorter leg.

(Fig. of Ex. 86. Draw CA to middle point of BD.)

105. If AC be drawn from the vertex of the right angle to the hypotenuse of right triangle BCD so as to make $\angle ACD = \angle D$, it bisects the hypotenuse.

(Fig. of Ex. 74. Prove $\triangle ABC$ isosceles.)



106. If D is the middle point of side BC of triangle ABC, prove $AD > \frac{1}{2}(AB + AC - BC)$. (§ 62.)

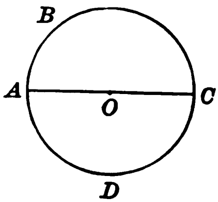
Note. For additional exercises on Book I., see p. 220.

Book II.

THE CIRCLE.

DEFINITIONS.

142. A *circle* (⊙) is a portion of a plane bounded by a curve called a *circumference*, all points of which are equally distant from a point within, called the *centre*; as $ABCD$.



An *arc* is any portion of the circumference; as AB.

A *radius* is a straight line drawn from the centre to the circumference; as OA.

A *diameter* is a straight line drawn through the centre, having its extremities in the circumference; as AC.

143. It follows from the definition of § 142 that

All radii of a circle are equal.

Also, all its diameters are equal, since each is the sum of two radii.

144. *Two circles are equal when their radii are equal.*

For they can evidently be applied one to the other so that their circumferences shall coincide throughout.

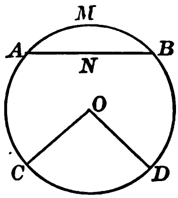
145. Conversely, *the radii of equal circles are equal.*

146. A *semi-circumference* is an arc equal to one-half the circumference.

A *quadrant* is an arc equal to one-fourth the circumference.

Concentric circles are circles having the same centre.

147. A *chord* is a straight line joining the extremities of an arc; as *AB*.



The arc is said to be *subtended* by its chord.

Every chord subtends two arcs; thus chord *AB* subtends arcs *AMB* and *ACDB*.

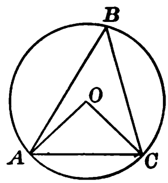
When the arc subtended by a chord is spoken of, that arc which is less than a semi-circumference is understood, unless the contrary is specified.

A *segment* of a circle is the portion included between an arc and its chord; as *AMBN*.

A *semicircle* is a segment equal to one-half the circle.

A *sector* of a circle is the portion included between an arc and the radii drawn to its extremities; as *OCD*.

148. A *central angle* is an angle whose vertex is at the centre, and whose sides are radii; as *AOC*.

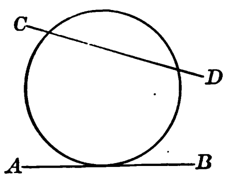


An *inscribed angle* is an angle whose vertex is on the circumference, and whose sides are chords; as *ABC*.

An angle is said to be *inscribed in a segment* when its vertex is on the arc of the segment, and its sides pass through the extremities of the subtending chord.

Thus, angle *B* is inscribed in segment *ABC*.

149. A straight line is said to be *tangent to*, or *touch, a circle* when it has but one point in common with the circumference; as *AB*.



In such a case, the circle is said to be tangent to the straight line.

The common point is called the *point of contact*, or *point of tangency*.

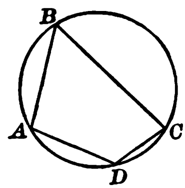
A *secant* is a straight line which intersects the circumference in two points; as *CD*.

150. Two circles are said to be ***tangent to each other*** when they are both tangent to the same straight line at the same point.

They are said to be tangent *internally* or *externally* according as one circle lies entirely within or entirely without the other.

A *common tangent* to two circles is a straight line which is tangent to both of them.

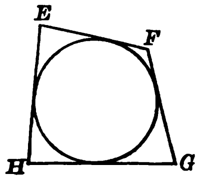
151. A polygon is said to be ***inscribed in a circle*** when all its vertices lie on the circumference; as *ABCD*.



In such a case, the circle is said to be *circumscribed about the polygon.*

A polygon is said to be ***inscriptible*** when it can be inscribed in a circle.

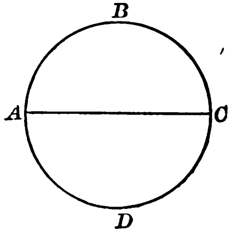
A polygon is said to be *circumscribed about a circle* when all its sides are tangent to the circle; as *EFGH*.



In such a case, the circle is said to be *inscribed in the polygon.*

PROP. I. THEOREM.

152. *Every diameter bisects the circle and its circumference.*



Given AC a diameter of ⊙ $ABCD$.

To Prove that AC bisects the ⊙, and its circumference.

Proof. Superpose segment *ABC* upon segment *ADC*, by folding it over about *AC* as an axis.

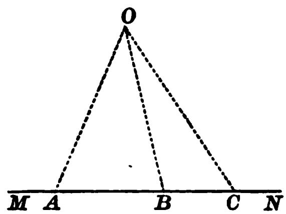
Then, arc *ABC* will coincide with arc *ADC*; for otherwise there would be points of the circumference unequally distant from the centre.

Hence, segments *ABC* and *ADC* coincide throughout, and are equal.

Therefore, *AC* bisects the ⊙, and its circumference.

Prop. II. Theorem.

153. *A straight line cannot intersect a circumference at more than two points.*



Given *O* the centre of a ⊙, and *MN* any str. line.

To Prove that *MN* cannot intersect the circumference at more than two points.

Proof. If possible, let *MN* intersect the circumference at three points, *A*, *B*, and *C*; draw radii *OA*, *OB*, and *OC*.

Then, $OA = OB = OC$. (§ 143)

We should then have three equal str. lines drawn from a point to a str. line.

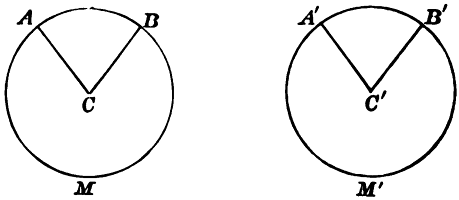
But this is impossible; for it follows from § 49 that not more than two equal str. lines can be drawn from a point to a str. line.

Hence, *MN* cannot intersect the circumference at more than two points.

Ex. 1. What is the locus of points at a given distance from a given point?

PROP. III. THEOREM.

154. *In equal circles, or in the same circle, equal central angles intercept equal arcs on the circumference.*



Given ACB and $A'C'B'$ equal central ∠s of equal ⊙s AMB and $A'M'B'$, respectively.

To Prove arc AB = arc $A'B'$.

Proof. Superpose sector ABC upon sector $A'B'C'$ in such a way that $\angle C$ shall coincide with its equal $\angle C'$.

Now, $AC = A'C'$ and $BC = B'C'$. (§ 145)

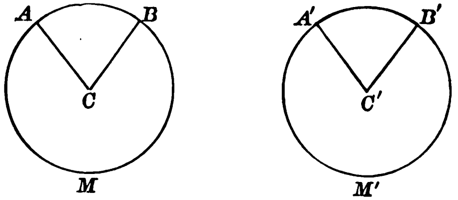
Whence, point A will fall at A', and point B at B'.

Then, arc AB will coincide with arc $A'B'$; for all points of either are equally distant from the centre.

$\therefore$ arc AB = arc $A'B'$.

PROP. IV. THEOREM.

155. (Converse of Prop. III.) *In equal circles, or in the same circle, equal arcs are intercepted by equal central angles.*



Given ACB and $A'C'B'$ central ∠s of equal ⊙s AMB and $A'M'B'$, respectively, and arc AB = arc $A'B'$.

To Prove $\angle C = \angle C'$.

Proof. Since the ⊙s are equal, we may superpose ⊙ AMB upon ⊙ $A'M'B'$ in such a way that point A shall fall at A', and centre C at C'.

Then since arc $AB =$ arc $A'B'$, point B will fall at B'.

Whence, radii AC and BC will coincide with radii $A'C'$ and $B'C'$, respectively. (Ax. 3)

Hence, $\angle C$ will coincide with $\angle C'$.

$$\therefore \angle C = \angle C'.$$

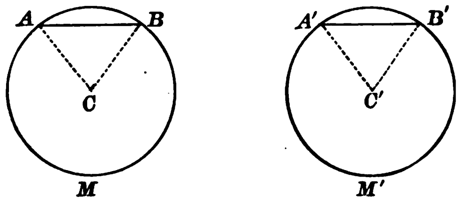
156. Sch. *In equal circles, or in the same circle,*

1. *The greater of two central angles intercepts the greater arc on the circumference.*

2. *The greater of two arcs is intercepted by the greater central angle.*

PROP. V. THEOREM.

157. *In equal circles, or in the same circle, equal chords subtend equal arcs.*



Given, in equal ⊙s AMB and $A'M'B'$,

chord $AB =$ chord $A'B'$.

To Prove arc $AB =$ arc $A'B'$.

Proof. Draw radii AC, BC, $A'C'$, and $B'C'$.

Then in △s ABC and $A'B'C'$, by hyp.,

$$AB = A'B'.$$

Also, $AC = A'C'$ and $BC = B'C'$. (?)

$\therefore \triangle ABC = \triangle A'B'C'$. (?)

$\therefore \angle C = \angle C'$. (?)

$\therefore$ arc $AB =$ arc $A'B'$. (§ 154)

PROP. VI. THEOREM.

158. (Converse of Prop. V.) *In equal circles, or in the same circle, equal arcs are subtended by equal chords.*

(Fig. of Prop. V.)

Given, in equal Ⓢ AMB and $A'M'B'$, arc AB = arc $A'B'$; and chords AB and $A'B'$.

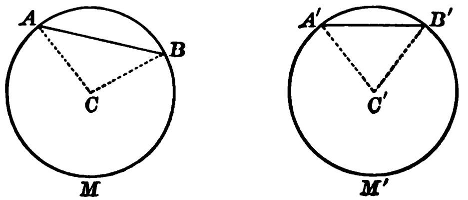
To Prove chord AB = chord $A'B'$.

(Prove $\triangle ABC = \triangle A'B'C'$, by § 63.)

Ex. 2. If two circumferences intersect each other, the distance between their centres is greater than the difference of their radii. (§ 62.)

PROP. VII. THEOREM.

159. *In equal circles, or in the same circle, the greater of two arcs is subtended by the greater chord; each arc being less than a semi-circumference.*



Given, in equal Ⓢ AMB and $A'M'B'$, arc $AB >$ arc $A'B'$, each arc being $<$ a semi-circumference, and chords AB and $A'B'$.

To Prove chord $AB >$ chord $A'B'$.

Proof. Draw radii AC, BC, $A'C'$, and $B'C'$.

Then in ⚠ ABC and $A'B'C'$,

$AC = A'C'$ and $BC = B'C'$. (?)

And since, by hyp., arc $AB >$ arc $A'B'$, we have

$\angle C > \angle C'$. (§ 156, 2)

$\therefore$ chord $AB >$ chord $A'B'$. (§ 91)

PROP. VIII. THEOREM.

160. (Converse of Prop. VII.) *In equal circles, or in the same circle, the greater of two chords subtends the greater arc; each arc being less than a semi-circumference.*

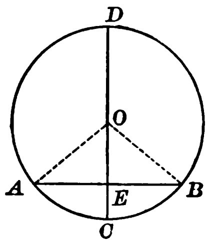
(Fig. of Prop. VII.)

($\angle C > \angle C'$, by § 92; the theorem follows by § 156, 1.)

161. Sch. If each arc is *greater* than a semi-circumference, the greater arc is subtended by the less chord; and conversely the greater chord subtends the less arc.

PROP. IX. THEOREM.

162. *The diameter perpendicular to a chord bisects the chord and its subtended arcs.*



Given, in $\odot ABD$, diameter $CD \perp$ chord AB.

To Prove that CD bisects chord AB, and arcs ACB and ADB.

Proof. Let O be the centre of the $\odot$, and draw radii OA and OB.

Then, $OA = OB$. (?)

Hence, $\triangle OAB$ is isosceles.

Therefore, CD bisects AB, and $\angle AOB$. (§ 94)

Then since $\angle AOC = \angle BOC$, we have

arc $AC =$ arc BC. (§ 154)

Again, $\angle AOD = \angle BOD$. (§ 31, 2)

$\therefore$ arc $AD =$ arc BD. (?)

Hence, CD bisects AB, and arcs ACB and ADB.

163. Cor. *The perpendicular erected at the middle point of a chord passes through the centre of the circle, and bisects the arcs subtended by the chord.*

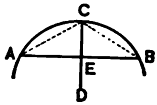
EXERCISES.

3. The diameter which bisects a chord is perpendicular to it and bisects its subtended arcs. (§ 43.)

(Fig. of Prop. IX. Given diameter CD bisecting chord AB.)

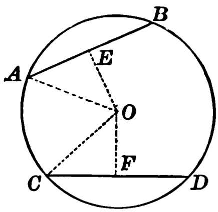
4. The straight line which bisects a chord and its subtended arc is perpendicular to the chord.

(By § 158, chord $AC =$ chord BC.)



PROP. X. THEOREM.

164. *In the same circle, or in equal circles, equal chords are equally distant from the centre.*



Given AB and CD equal chords of $\odot$ ABC, whose centre is O, and lines OE and $OF \perp$ to AB and CD, respectively.

To Prove $OE = OF$. (§ 47)

Proof. Draw radii OA and OC.

Then in rt. $\triangle$ OAE and OCF,

$OA = OC$. (?)

Now, E is the middle point of AB, and F of CD. (§ 162)

$\therefore AE = CF$,

being halves of equal chords AB and CD, respectively.

$\therefore \triangle OAE = \triangle OCF$. (?)

$\therefore OE = OF$. (?)

PROP. XI. THEOREM.

165. (Converse of Prop. X.) *In the same circle, or in equal circles, chords equally distant from the centre are equal.*

(Fig. of Prop. X.)

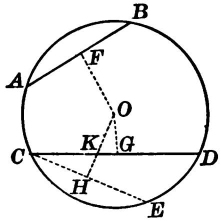
Given O the centre of $\odot ABC$, and AB and CD chords equally distant from O.

To Prove chord $AB =$ chord CD.

(Rt. $\triangle OAE =$ rt. $\triangle OCF$, and $AE = CF$; E is the middle point of AB, and F of CD.)

PROP. XII. THEOREM.

166. *In the same circle, or in equal circles, the less of two chords is at the greater distance from the centre.*



Given, in $\odot ABC$, chord $AB <$ chord CD, and $\perp$s OF and OG drawn from centre O to AB and CD, respectively.

To Prove $OF > OG$.

Proof. Since chord $AB <$ chord CD, we have

arc $AB <$ arc CD. (§ 160)

Lay off arc $CE =$ arc AB, and draw line CE.

$\therefore$ chord $CE =$ chord AB. (§ 158)

Draw line $OH \perp CE$, intersecting CD at K.

$\therefore OH = OF$. (§ 164)

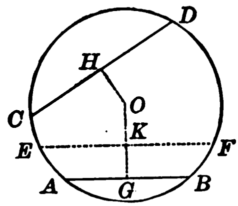
But, $OH > OK$.

And, $OK > OG$. (?)

Whence, OH, or its equal OF, is $> OG$.

PROP. XIII. THEOREM.

167. (Converse of Prop. XII.) *In the same circle, or in equal circles, if two chords are unequally distant from the centre, the more remote is the less.*



Given O the centre of $\odot ABC$, and chord AB more remote from O than chord CD.

To Prove chord $AB <$ chord CD.

Proof. Draw lines OG and $OH \perp$ to AB and CD respectively, and on OG lay off $OK = OH$.

Through K draw chord $EF \perp OK$.

$\therefore$ chord $EF =$ chord CD. (§ 165)

Now, chord $AB \parallel$ chord EF. (§ 54)

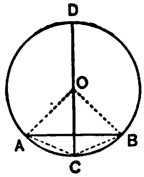
Then it is evident that arc AB is $<$ arc EF, for it is only a portion of arc EF.

$\therefore$ chord $AB <$ chord EF. (§ 159)

$\therefore$ chord $AB <$ chord CD.

168. Cor. *A diameter of a circle is greater than any other chord;* for a chord which passes through the centre is greater than any chord which does not. (§ 167)

EXERCISES.

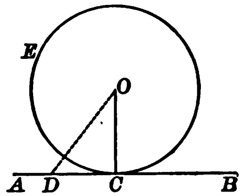


5. The diameter which bisects an arc bisects its chord at right angles.

6. The perpendiculars to the sides of an inscribed quadrilateral at their middle points meet in a common point. (§ 163.)

PROP. XIV. THEOREM.

169. *A straight line perpendicular to a radius of a circle at its extremity is tangent to the circle.*



Given line $AB \perp$ to radius OC of $\odot EC$ at C.

To Prove AB tangent to the $\odot$.

Proof. Let D be any point of AB except C, and draw line OD.

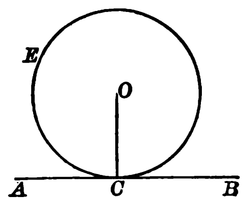
$$\therefore OD > OC. \qquad (?)$$

Therefore, point D lies without the $\odot$.

Then, every point of AB except C lies without the $\odot$, and AB is tangent to the $\odot$. (§ 149)

PROP. XV. THEOREM.

170. (Converse of Prop. XIV.) *A tangent to a circle is perpendicular to the radius drawn to the point of contact.*



Given line AB tangent to $\odot EC$ at C, and radius OC.

To Prove $OC \perp AB$.

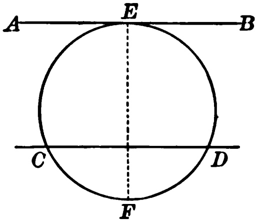
(OC is the shortest line that can be drawn from O to AB.)

171. Cor. *A line perpendicular to a tangent at its point of contact passes through the centre of the circle.*

Prop. XVI. Theorem.

172. *Two parallels intercept equal arcs on a circumference.*

Case I. *When one line is a tangent and the other a secant.*



Given AB a tangent to ⊙ CED at E, and CD a secant ∥ AB, intersecting the circumference at C and D.

To Prove arc CE = arc DE.

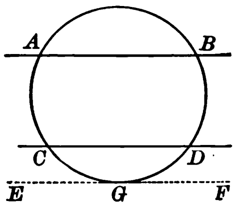
Proof. Draw diameter EF.

$\therefore EF \perp AB$. (§ 170)

$\therefore EF \perp CD$. (?)

$\therefore$ arc CE = arc DE. (§ 162)

Case II. *When both lines are secants.*



Given, in ⊙ ABC, AB and CD ∥ secants, intersecting the circumference at A and B, and C and D, respectively.

To Prove arc AC = arc BD.

Proof. Draw tangent $EF \parallel AB$, touching the ⊙ at G.

$\therefore EF \parallel CD$. (?)

Now, arc AG = arc BG,

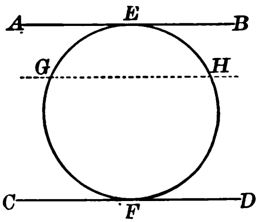
and arc CG = arc DG. (§ 172, Case I)

Subtracting, we have

$$\text{arc } AG - \text{arc } CG = \text{arc } BG - \text{arc } DG.$$

$$\therefore \text{arc } AC = \text{arc } BD.$$

Case III. *When both lines are tangents.*



Given, in ⊙ EGF, AB and CD ∥ tangents, touching the ⊙ at E and F, respectively.

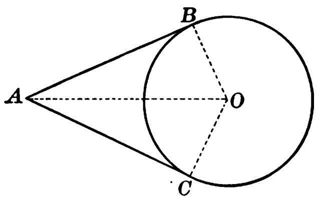
To Prove arc EGF = arc EHF.

(Draw secant GH ∥ AB.)

173. Cor. *The straight line joining the points of contact of two parallel tangents is a diameter.*

Prop. XVII. Theorem.

174. *The tangents to a circle from an outside point are equal.*



(Rt. △ OAB = rt. △ OAC, by § 90; then $AB = AC$.)

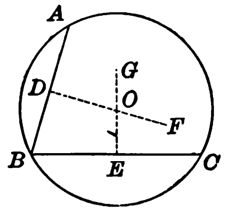
175. Cor. From equal ⊿s OAB and OAC,

$$\angle OAB = \angle OAC \text{ and } \angle AOB = \angle AOC.$$

Then, *the line joining the centre of a circle to the point of intersection of two tangents makes equal angles with the tangents, and also with the radii drawn to the points of contact.*

PROP. XVIII. THEOREM.

176. *Through three points, not in the same straight line, a circumference can be drawn, and but one.*



Given points A, B, and C, not in the same straight line.

To Prove that a circumference can be drawn through A, B, and C, and but one.

Proof. Draw lines AB and BC, and lines DF and $EG \perp$ to AB and BC, respectively, at their middle points, meeting at O.

Then O is equally distant from A, B, and C. (§ 137)

Hence, a circumference described with O as a centre and OA as a radius will pass through A, B, and C.

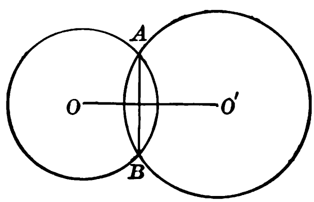
Again, the centre of any circumference drawn through A, B, and C must be in each of the $\perp$s DF and EG. (§ 42)

Then as DF and EG intersect in but one point, only one circumference can be drawn through A, B, and C.

177. Cor. *Two circumferences can intersect in but two points;* for if they had three common points, they would have the same centre, and coincide throughout.

PROP. XIX. THEOREM.

178. *If two circumferences intersect, the straight line joining their centres bisects their common chord at right angles.*



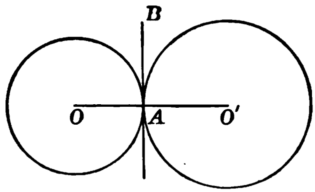
Given O and O' the centres of two ⓢ, whose circumferences intersect at A and B, and lines OO' and AB.

To Prove that OO' bisects AB at rt. ∡s.

(The proposition follows by § 43.)

Prop. XX. Theorem.

179. *If two circles are tangent to each other, the straight line joining their centres passes through their point of contact.*

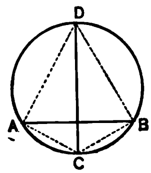


Given O and O' the centres of two ⓢ, which are tangent to line AB at A.

To Prove that str. line joining O and O' passes through A.

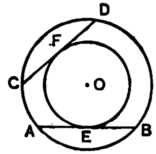
(Draw radii OA and $O'A$; since these lines are $\perp AB$, OAO' is a str. line by § 37; the proposition follows by Ax. 3.)

EXERCISES.



7. The straight line which bisects the arcs subtended by a chord bisects the chord at right angles.

8. The tangents to a circle at the extremities of a diameter are parallel.



9. If two circles are concentric, any two chords of the greater which are tangent to the less are equal. (§ 165.)

10. The straight line drawn from the centre of a circle to the point of intersection of two tangents bisects at right angles the chord joining their points of contact. (§ 174.)

ON MEASUREMENT.

180. The *ratio* of a magnitude to another of the same kind is the quotient of the first divided by the second.

Thus, if a and b are quantities of the same kind, the ratio of a to b is $\frac{a}{b}$; it may also be expressed $a : b$.

A magnitude is *measured* by finding its ratio to another magnitude of the same kind, called the *unit of measure.*

The quotient, if it can be obtained exactly as an integer or fraction, is called the *numerical measure* of the magnitude.

181. Two magnitudes of the same kind are said to be *commensurable* when a unit of measure, called a *common measure,* is contained an *integral* number of times in each.

Thus, two lines whose lengths are $2\frac{3}{4}$ and $3\frac{4}{5}$ inches are commensurable; for the common measure $\frac{1}{20}$ inch is contained an integral number of times in each; i.e., 55 times in the first line, and 76 times in the second.

Two magnitudes of the same kind are said to be *incommensurable* when no magnitude of the same kind can be found which is contained an integral number of times in each.

For example, let AB and CD be two lines such that

$$\frac{AB}{CD} = \sqrt{2}.$$

As $\sqrt{2}$ can only be obtained *approximately,* no line, *however small,* can be found which is contained an integral number of times in each line, and AB and CD are incommensurable.

182. A magnitude which is incommensurable with respect to the unit has, strictly speaking, no numerical measure (§ 180); still if CD is the unit of measure, and $\frac{AB}{CD} = \sqrt{2}$, we shall speak of $\sqrt{2}$ as the numerical measure of AB.

183. It is evident from the above that the ratio of two magnitudes of the same kind, whether commensurable or incommensurable, is equal to the ratio of their numerical measures when referred to a common unit.

THE METHOD OF LIMITS.

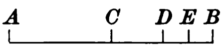
184. A *variable quantity*, or simply a *variable*, is a quantity which may assume, under the conditions imposed upon it, an indefinitely great number of *different* values.

185. A *constant* is a quantity which remains unchanged throughout the same discussion.

186. A *limit* of a variable is a constant quantity, the difference between which and the variable may be made less than any assigned quantity, however small, but cannot be made equal to zero.

In other words, a limit of a variable is a fixed quantity to which the variable approaches indefinitely near, but never actually reaches.

187. Suppose, for example, that a point moves from A towards B under the condition that it shall move, during successive equal intervals of time, first from A to C, half-way between A and B; then to D, half-way between C and B; then to E, half-way between D and B; and so on indefinitely.



In this case, the distance between the moving point and B can be made less than any assigned distance, however small, but cannot be made equal to 0.

Hence, the distance from A to the moving point is a variable which approaches the constant distance AB as a limit.

Again, the distance from the moving point to B is a variable which approaches the limit 0.

As another illustration, consider the series

$$1, \tfrac{1}{2}, \tfrac{1}{4}, \tfrac{1}{8}, \tfrac{1}{16}, \cdots,$$

where each term after the first is one-half the preceding.

In this case, by taking terms enough, the last term may be made less than any assigned number, however small, but cannot be made actually equal to 0.

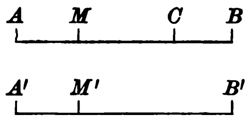
Then, the last term of the series is a variable which approaches the limit 0 when the number of terms is indefinitely increased.

Again, the sum of the first two terms is $1\frac{1}{2}$;
the sum of the first three terms is $1\frac{3}{4}$;
the sum of the first four terms is $1\frac{7}{8}$; etc.

In this case, by taking terms enough, the sum of the terms may be made to differ from 2 by less than any assigned number, however small, but cannot be made actually equal to 2.

Then, the sum of the terms of the series is a variable which approaches the limit 2 when the number of terms is indefinitely increased.

188. The Theorem of Limits. *If two variables are always equal, and each approaches a limit, the limits are equal.*



Given AM and $A'M'$ two variables, which are always equal, and approach the limits AB and $A'B'$, respectively.

To Prove $AB = A'B'$.

Proof. If possible, let AB be $> A'B'$; and lay off, on AB, $AC = A'B'$.

Then, variable AM may have values $> AC$, while variable $A'M'$ is restricted to values $< AC$; which is contrary to the hypothesis that the variables are always equal.

Hence, AB cannot be $> A'B'$.

In like manner, it may be proved that AB cannot be $< A'B'$.

Therefore, since AB can be neither $>$, nor $< A'B'$, we have

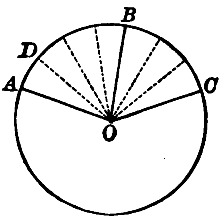
$$AB = A'B'.$$

MEASUREMENT OF ANGLES.

PROP. XXI. THEOREM.

189. *In the same circle, or in equal circles, two central angles are in the same ratio as their intercepted arcs.*

Case I. *When the arcs are commensurable* (§ 181).



Given, in ⊙ ABC, AOB and BOC central ∠s intercepting commensurable arcs AB and BC, respectively.

To Prove $\frac{\angle AOB}{\angle BOC} = \frac{\text{arc } AB}{\text{arc } BC}.$

Proof. Since, by hyp., arcs AB and BC are commensurable, let arc AD be a common measure of arcs AB and BC; and suppose it to be contained 4 times in arc AB, and 3 times in arc BC.

$$\therefore \frac{\text{arc } AB}{\text{arc } BC} = \frac{4}{3}. \quad (1)$$

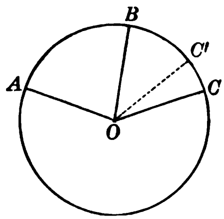
Drawing radii to the several points of division of arc AC, ∠ AOB will be divided into 4 ∠s, and ∠ BOC into 3 ∠s, all of which ∠s are equal. (§ 155)

$$\therefore \frac{\angle AOB}{\angle BOC} = \frac{4}{3}. \quad (2)$$

From (1) and (2), we have

$$\frac{\angle AOB}{\angle BOC} = \frac{\text{arc } AB}{\text{arc } BC}. \quad (?)$$

Case II. *When the arcs are incommensurable* (§ 181).



Given, in ⊙ ABC, AOB and BOC central ∠s intercepting incommensurable arcs AB and BC, respectively.

To Prove $\frac{\angle AOB}{\angle BOC} = \frac{\text{arc } AB}{\text{arc } BC}$.

Proof. Let arc AB be divided into any number of equal arcs, and let one of these arcs be applied to arc BC as a unit of measure.

Since arcs AB and BC are incommensurable, a certain number of the equal arcs will extend from B to C', leaving a remainder $C'C$ less than one of the equal arcs.

Draw radius OC'.

Then, since by const., arcs AB and BC' are commensurable,

$$\frac{\angle AOB}{\angle BOC'} = \frac{\text{arc } AB}{\text{arc } BC'}. \quad (\text{§ 189, Case I.})$$

Now let the number of subdivisions of arc AB be indefinitely increased.

Then the unit of measure will be indefinitely diminished; and the remainder $C'C$, being always less than the unit, will approach the limit 0.

Then $\angle BOC'$ will approach the limit $\angle BOC$,

and arc BC' will approach the limit arc BC.

Hence, $\frac{\angle AOB}{\angle BOC'}$ will approach the limit $\frac{\angle AOB}{\angle BOC}$,

and $\frac{\text{arc } AB}{\text{arc } BC'}$ will approach the limit $\frac{\text{arc } AB}{\text{arc } BC}$.

Now, $\frac{\angle AOB}{\angle BOC'}$ and $\frac{\text{arc } AB}{\text{arc } BC'}$ are variables which are always equal, and approach the limits $\frac{\angle AOB}{\angle BOC}$ and $\frac{\text{arc } AB}{\text{arc } BC}$, respectively.

By the Theorem of Limits, these limits are equal. (§ 188)

$$\therefore \frac{\angle AOB}{\angle BOC} = \frac{\text{arc } AB}{\text{arc } BC}.$$

190. Sch. The usual unit of measure for arcs is the *degree,* which is the ninetieth part of a quadrant (§ 146).

The degree of arc is divided into sixty equal parts, called *minutes,* and the minute into sixty equal parts, called *seconds.*

If the sum of two arcs is a quadrant, or 90°, one is called the *complement* of the other; if their sum is a semi-circumference, or 180°, one is called the *supplement* of the other.

191. Cor. I. By § 154, equal central ∠s, in the same ⊙, intercept equal arcs on the circumference.

Hence, if the angular magnitude about the centre of a ⊙ be divided into four equal ∠s, each ∠ will intercept an arc equal to one-fourth of the circumference.

That is, *a right central angle intercepts a quadrant on the circumference.* (§ 35)

192. Cor. II. By § 189, a central ∠ of n degrees bears the same ratio to a rt. central ∠ that its intercepted arc bears to a quadrant.

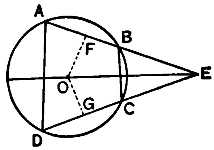
But a central ∠ of n degrees is $\frac{n}{90}$ of a rt. central ∠.

Hence, its intercepted arc is $\frac{n}{90}$ of a quadrant, or an arc of n degrees.

The above principle is usually expressed as follows:

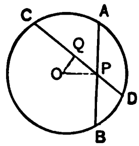
A central angle is measured by its intercepted arc.

This means simply that the number of angular degrees in a central angle is equal to the number of degrees of arc in its intercepted arc.



66. If sides AB and CD of inscribed quadrilateral $ABCD$ make equal angles with the diameter passing through their point of intersection, prove $AB = CD$. (§ 165.)

67. If angles A, B, and C of circumscribed quadrilateral $ABCD$ are 128°, 67°, and 112°, respectively, and sides AB, BC, CD, and DA are tangent to the circle at points E, F, G, and H, respectively, find the number of degrees in each angle of quadrilateral $EFGH$.



68. The chord drawn through a given point within a circle, perpendicular to the diameter passing through the point, is the least chord which can be drawn through the given point. (§ 165.)

(Given chords AB and CD drawn through P, and $AB \perp OP$. To prove $AB < CD$.)

69. If ADB, BEC, and CFA are angles inscribed in segments ABD, BCE, and ACF, respectively, exterior to inscribed triangle ABC, prove their sum equal to four right angles. (§ 196.)

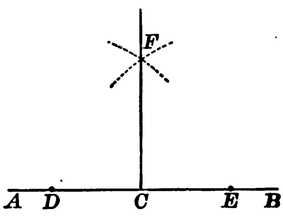
Note. For additional exercises on Book II., see p. 222.

CONSTRUCTIONS.

Prop. XXVII. Problem.

203. *At a given point in a straight line to erect a perpendicular to that line.*

First Method.



Given C any point in line AB.

Required to draw a line ⊥ to AB at C.

Construction. Take points D and E on AB equally distant from C.

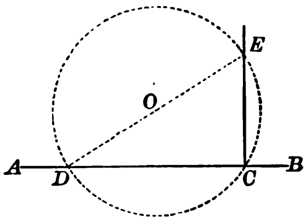
With D and E as centres, and with equal radii, describe arcs intersecting at F, and draw line CF.

Then, CF is ⊥ to AB at C.

Proof. By cons., C and F are each equally distant from D and E.

Whence, CF is ⊥ to DE at its middle point. (?)

Second Method.



Given C any point in line AB.

Required to draw a line ⊥ to AB at C.

Construction. With any point O without line AB as a centre, and distance OC as a radius, describe a circumference intersecting AB at C and D.

Draw diameter DE, and line CE.

Then, CE is ⊥ to AB at C.

Proof. ∠ DCE, being inscribed in a semicircle, is a rt. ∠. (§ 195)

$$\therefore CE \perp CD.$$

Note. The second method of construction is preferable when the given point is near the end of the line.

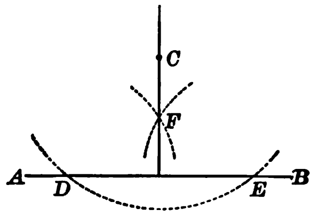
EXERCISES.

70. Given the base and altitude of an isosceles triangle, to construct the triangle.

71. Given an acute angle, to construct its complement.

Prop. XXVIII. Problem.

204. *From a given point without a straight line to draw a perpendicular to that line.*



Given C any point without line AB.

Required to draw from C a line ⊥ to AB.

Construction. With C as a centre, and any convenient radius, describe an arc intersecting AB at D and E.

With D and E as centres, and with equal radii, describe arcs intersecting at F.

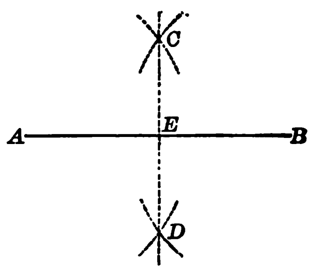
Draw line CF.

Then, $CF \perp AB$.

Proof. Since, by cons., C and F are each equally distant from D and E, CF is ⊥ to DE at its middle point. (?)

Prop. XXIX. Problem.

205. *To bisect a given straight line.*



Given line AB.

Required to bisect AB.

Construction. With A and B as centres, and with equal radii, describe arcs intersecting at C and D.

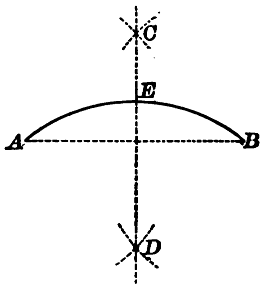
Draw line CD intersecting AB at E.

Then, E is the middle point of AB.

(The proof is left to the pupil.)

Prop. XXX. Problem.

206. *To bisect a given arc.*



Given arc AB.

Required to bisect arc AB.

Construction. With A and B as centres, and with equal radii, describe arcs intersecting at C and D.

Draw line CD intersecting arc AB at E.

Then E is the middle point of arc AB.

Proof. Draw chord AB.

Then, CD is $\perp$ to chord AB at its middle point. (?)

Whence, CD bisects arc AB. (§ 163)

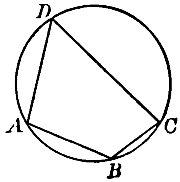
EXERCISES.

72. Given an angle, to construct its supplement.

73. Given a side of an equilateral triangle, to construct the triangle.

74. To construct an angle of 60° (Ex. 73); of 30° (Ex. 71).

75. To construct an angle of 120° (Ex. 72); of 150°.



200. Cor. (Converse of § 196.) *If the opposite angles of a quadrilateral are supplementary, the quadrilateral can be inscribed in a circle.*

Given, in quadrilateral $ABCD$, $\angle A$ sup. to $\angle C$, and $\angle B$ to $\angle D$; also, a circumference drawn through A, B, and C. (§ 176)

To Prove that D lies on the circumference.

Proof. Since $\angle D$ is sup. to $\angle B$, it is measured by $\frac{1}{2}$ arc ABC. (§ 193)

Then, D must lie on the circumference; for if it were within the ⊙, $\angle D$ would be measured by $\frac{1}{2}$ an arc $> ABC$; and if it were without the ⊙, $\angle D$ would be measured by $\frac{1}{2}$ an arc $< ABC$. (§§ 198, 199)

Prop. XXVI. Theorem.

201. *The angle between a secant and a tangent, or two tangents, is measured by one-half the difference of the intercepted arcs.*

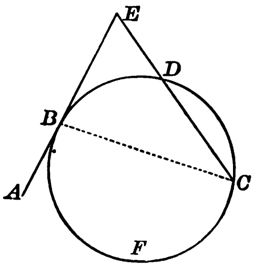


Fig. 1.

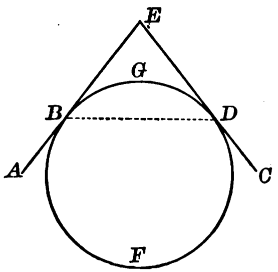


Fig. 2.

1. **Given** AE a tangent to ⊙ BDC at B, and EC a secant intersecting the circumference at C and D. (Fig. 1.)

To Prove that $\angle E$ is measured by $\frac{1}{2}$ (arc BFC − arc BD). (We have $\angle E = \angle ABC - \angle C$.)

2. (In Fig. 2, $\angle E = \angle ABD - \angle BDE$; then use § 197.)

202. Cor. Since a circumference is an arc of 360°, we have

$\frac{1}{2}$ (arc BFD – arc BGD)

$$= \tfrac{1}{2}(360° - \text{arc } BGD - \text{arc } BGD)$$
$$= \tfrac{1}{2}(360° - 2 \text{ arc } BGD)$$
$$= 180° - \text{arc } BGD.$$

Then, $\angle E$ is measured by 180° – arc BGD.

Hence, *the angle between two tangents is measured by the supplement of the smaller of the two intercepted arcs.*

EXERCISES.

11. If, in figure of § 197, arc $BC = 107°$, how many degrees are there in angles ABC and EBC?

12. If, in figure of § 198, arc $AC = 74°$, and $\angle AEC = 51°$, how many degrees are there in arc BD?

13. If, in figure of § 199, arc $AC = 117°$, and $\angle C = 14°$, how many degrees are there in angle E?

14. If, in figure of § 199, AC is a quadrant, and $\angle E = 39°$, how many degrees are there in arc BD?

15. If, in Fig. 1 of § 201, arc $BFC = 197°$, and arc $CD = 75°$, how many degrees are there in angle E?

16. If, in Fig. 1 of § 201, $\angle E = 53°$, and arc BD is one-fifth of the circumference, how many degrees are there in arc BFC?

17. If, in Fig. 2 of § 201, arc BFD is thirteen-sixteenths of the circumference, how many degrees are there in angle E?

18. Three consecutive sides of an inscribed quadrilateral subtend arcs of 82°, 99°, and 67° respectively. Find each angle of the quadrilateral in degrees, and the angle between its diagonals.

19. Prove Prop. XXIV. by drawing through B a chord parallel to CD. (§ 172.)

20. Prove Prop. XXV. by drawing through B a chord parallel to CD.

21. Prove Prop. XXVI. for Fig. 1 by drawing through D a chord parallel to AE.

22. An angle inscribed in a segment greater than a semicircle is acute; and an angle inscribed in a segment less than a semicircle is obtuse. (§ 193.)

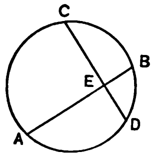
23. In an inscribed trapezoid the non-parallel sides are equal, and also the diagonals.

(The non-parallel sides, and also the diagonals, subtend equal arcs.)

24. If the inscribed and circumscribed circles of a triangle are concentric, prove the triangle equilateral. (§ 165.)

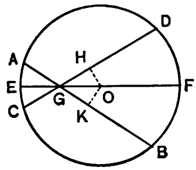
25. If AB and AC are the tangents from point A to the circle whose centre is O, prove $\angle BAC = 2 \angle OBC$. (Ex. 10, p. 87.)

26. If two chords intersect at right angles within the circumference of a circle, the sum of the opposite intercepted arcs is equal to a semi-circumference.

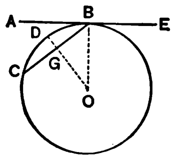


27. Two intersecting chords which make equal angles with the diameter passing through their point of intersection are equal. (§ 165.)

(Prove that $OH = OK$.)



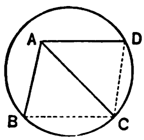
28. Prove Prop. XXIII. by drawing a radius perpendicular to BC. (§ 162.)



29. If AB and AC are two chords of a circle making equal angles with the tangent at A, prove $AB = AC$.

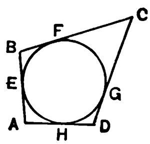
30. From a given point within a circle and not coincident with the centre, not more than two equal straight lines can be drawn to the circumference.

(If possible, let AB, AC, and AD be three equal straight lines from point A to circumference BCD; then, by § 163, A must coincide with the centre.)



31. The sum of two opposite sides of a circumscribed quadrilateral is equal to the sum of the other two sides. (§ 174.)

(To prove $AB + CD = AD + BC$.)



32. Prove Prop. VI. by superposition.

33. In a circumscribed trapezoid, the straight line joining the middle points of the non-parallel sides is equal to one-fourth the perimeter of the trapezoid. (§ 132.)

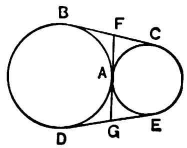
34. If the opposite sides of a circumscribed quadrilateral are parallel, the figure is a rhombus or a square. (Ex. 31.)

(Prove the sides all equal.)

35. If tangents be drawn to a circle at the extremities of any pair of diameters which are not perpendicular to each other, the figure formed is a rhombus. (Ex. 34.)

36. If the angles of a circumscribed quadrilateral are right angles, the figure is a square.

37. If two circles are tangent to each other at point *A*, the tangents to them from any point in the common tangent which passes through *A* are equal. (§ 174.)



38. If two circles are tangent to each other externally at point *A*, the common tangent which passes through *A* bisects the other two common tangents. (Ex. 37.)

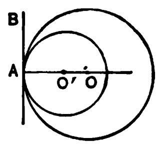
(To prove that *FG* bisects *BC* and *DE*.)

39. The bisector of the angle between two tangents to a circle passes through the centre.

(The bisector of the ∠ between the tangents bisects at rt. ∠s the chord joining their points of contact.)

40. The bisectors of the angles of a circumscribed quadrilateral pass through a common point.

41. If *AB* is one of the non-parallel sides of a trapezoid circumscribed about a circle whose centre is *O*, prove *AOB* a right angle. (§ 175.)



42. If two circles are tangent to each other internally, the distance between their centres is equal to the difference of their radii.

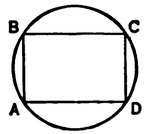
43. Prove the theorem of § 168 by drawing radii to the extremities of the chord. (Ax. 4.)

44. Prove the theorem of § 202 by drawing radii to the points of contact of the tangents. (§ 192.)

45. If any number of angles are inscribed in the same segment, their bisectors pass through a common point. (§ 193.)

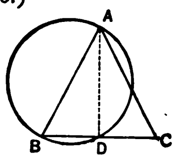
46. Prove Prop. XIII. by *Reductio ad Absurdum.* (§§ 164, 166.)

47. Two chords perpendicular to a third chord at its extremities are equal. (§ 158.)

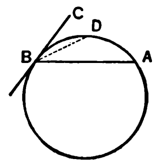


48. If two opposite sides of an inscribed quadrilateral are equal and parallel, the figure is a rectangle. (Arc BCD is a semi-circumference.)

49. If the diagonals of an inscribed quadrilateral intersect at the centre of the circle, the figure is a rectangle. (§ 195.)



50. The circle described on one of the equal sides of an isosceles triangle as a diameter, bisects the base. (§ 195.)



51. If a tangent be drawn to a circle at the extremity of a chord, the middle point of the subtended arc is equally distant from the chord and from the tangent.
(BD bisects $\angle ABC$.)

52. If sides AB, BC, and CD of an inscribed quadrilateral subtend arcs of 99°, 106°, and 78°, respectively, and sides BA and CD produced meet at E, and sides AD and BC at F, find the number of degrees in angles AED and AFB.

53. If O is the centre of the circumscribed circle of triangle ABC, and OD be drawn perpendicular to BC, prove

$$\angle BOD = \angle A. \quad (\S\ 192.)$$

54. If D, E, and F are the points of contact of sides AB, BC, and CA respectively of a triangle circumscribed about a circle, prove

$$\angle DEF = 90° - \tfrac{1}{2} A. \quad (\S\ 202.)$$

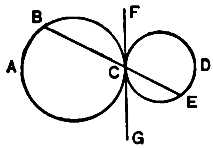
55. If sides AB and BC of an inscribed quadrilateral $ABCD$ subtend arcs of 69° and 112°, respectively, and angle AED between the diagonals is 87°, how many degrees are there in each angle of the quadrilateral?

56. If any number of parallel chords be drawn in a circle, their middle points lie in the same straight line. (§ 162.)

57. What is the locus of the middle points of a system of parallel chords in a circle?

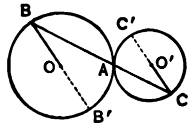
58. What is the locus of the middle points of a system of chords of given length in a circle?

59. If two circles are tangent to each other, any straight line drawn through their point of contact subtends arcs of the same number of degrees on their circumferences. (§ 197.)



(Let the pupil draw the figure for the case when the Ⓢ are tangent internally.)

60. If a straight line be drawn through the point of contact of two circles which are tangent to each other externally, terminating in their circumferences, the radii drawn to its extremities are parallel. (§ 73.)



(Let the pupil state the corresponding theorem for the case when the Ⓢ are tangent internally.)

61. If a straight line be drawn through the point of contact of two circles which are tangent to each other externally, terminating in their circumferences, the tangents at its extremities are parallel. (§ 73.)

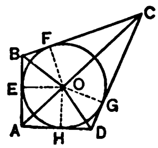
(Let the pupil state the corresponding theorem for the case when the Ⓢ are tangent internally.)

62. If sides AB and DC of inscribed quadrilateral $ABCD$ be produced to meet at E, prove that triangles ACE and BDE, and also triangles ADE and BCE, are mutually equiangular.

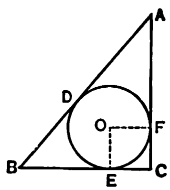
(For second part, see § 196.)

63. The sum of the angles subtended at the centre of a circle by two opposite sides of a circumscribed quadrilateral is equal to two right angles. (§ 175.)

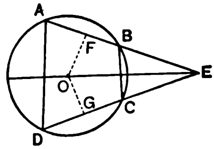
(To prove $\angle AOB + \angle COD = 180°$.)



64. If a circle is inscribed in a right triangle, the sum of its diameter and the hypotenuse is equal to the sum of the legs. (§ 174.)

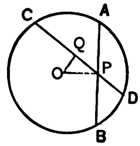


65. If a circle be described on the radius of another circle as a diameter, any chord of the greater passing through the point of contact of the circles is bisected by the circumference of the smaller. (§ 195.)



66. If sides AB and CD of inscribed quadrilateral $ABCD$ make equal angles with the diameter passing through their point of intersection, prove $AB = CD$. (§ 165.)

67. If angles A, B, and C of circumscribed quadrilateral $ABCD$ are 128°, 67°, and 112°, respectively, and sides AB, BC, CD, and DA are tangent to the circle at points E, F, G, and H, respectively, find the number of degrees in each angle of quadrilateral $EFGH$.



68. The chord drawn through a given point within a circle, perpendicular to the diameter passing through the point, is the least chord which can be drawn through the given point. (§ 165.)

(Given chords AB and CD drawn through P, and $AB \perp OP$. To prove $AB < CD$.)

69. If ADB, BEC, and CFA are angles inscribed in segments ABD, BCE, and ACF, respectively, exterior to inscribed triangle ABC, prove their sum equal to four right angles. (§ 196.)

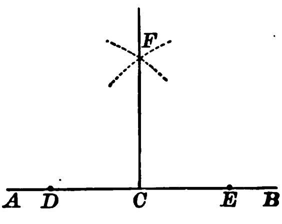
Note. For additional exercises on Book II., see p. 222.

CONSTRUCTIONS.

Prop. XXVII. Problem.

203. *At a given point in a straight line to erect a perpendicular to that line.*

First Method.



Given C any point in line AB.

Required to draw a line ⊥ to *AB* at *C*.

Construction. Take points *D* and *E* on *AB* equally distant from *C*.

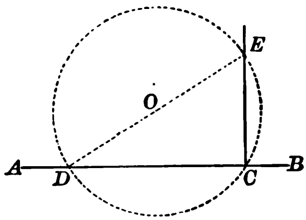
With *D* and *E* as centres, and with equal radii, describe arcs intersecting at *F*, and draw line *CF*.

Then, *CF* is ⊥ to *AB* at *C*.

Proof. By cons., *C* and *F* are each equally distant from *D* and *E*.

Whence, *CF* is ⊥ to *DE* at its middle point. (?)

Second Method.



Given *C* any point in line *AB*.

Required to draw a line ⊥ to *AB* at *C*.

Construction. With any point *O* without line *AB* as a centre, and distance *OC* as a radius, describe a circumference intersecting *AB* at *C* and *D*.

Draw diameter *DE*, and line *CE*.

Then, *CE* is ⊥ to *AB* at *C*.

Proof. ∠ *DCE*, being inscribed in a semicircle, is a rt. ∠. (§ 195)

$$\therefore CE \perp CD.$$

Note. The second method of construction is preferable when the given point is near the end of the line.

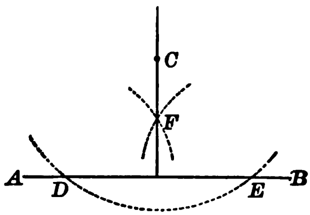
EXERCISES.

70. Given the base and altitude of an isosceles triangle, to construct the triangle.

71. Given an acute angle, to construct its complement.

PROP. XXVIII. PROBLEM.

204. *From a given point without a straight line to draw a perpendicular to that line.*



Given C any point without line AB.

Required to draw from C a line ⊥ to AB.

Construction. With C as a centre, and any convenient radius, describe an arc intersecting AB at D and E.

With D and E as centres, and with equal radii, describe arcs intersecting at F.

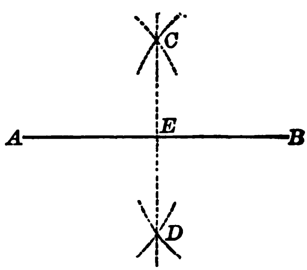
Draw line CF.

Then, $CF \perp AB$.

Proof. Since, by cons., C and F are each equally distant from D and E, CF is ⊥ to DE at its middle point. (?)

PROP. XXIX. PROBLEM.

205. *To bisect a given straight line.*



Given line AB.

Required to bisect AB.

Construction. With A and B as centres, and with equal radii, describe arcs intersecting at C and D.

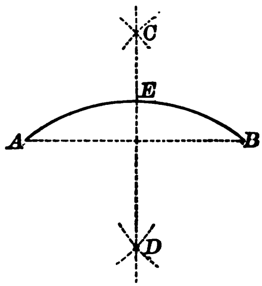
Draw line CD intersecting AB at E.

Then, E is the middle point of AB.

(The proof is left to the pupil.)

Prop. XXX. Problem.

206. *To bisect a given arc.*



Given arc AB.

Required to bisect arc AB.

Construction. With A and B as centres, and with equal radii, describe arcs intersecting at C and D.

Draw line CD intersecting arc AB at E.

Then E is the middle point of arc AB.

Proof. Draw chord AB.

Then, CD is $\perp$ to chord AB at its middle point. (?)

Whence, CD bisects arc AB. (§ 163)

EXERCISES.

72. Given an angle, to construct its supplement.

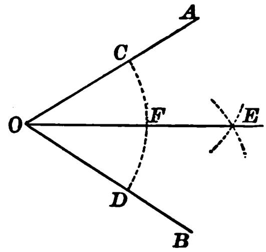
73. Given a side of an equilateral triangle, to construct the triangle.

74. To construct an angle of 60° (Ex. 73); of 30° (Ex. 71).

75. To construct an angle of 120° (Ex. 72); of 150°.

PROP. XXXI. PROBLEM.

207. *To bisect a given angle.*



Given $\angle AOB$.

Required to bisect $\angle AOB$.

Construction. With O as a centre, and any convenient radius, describe an arc intersecting OA at C, and OB at D.

With C and D as centres, and with the same radius as before, describe arcs intersecting at E, and draw line OE.

Then, OE bisects $\angle AOB$.

Proof. Let OE intersect arc CD at F.

By cons., O and E are each equally distant from C and D.

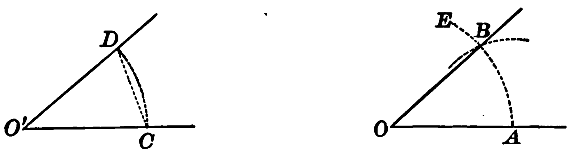
Whence, OE bisects arc CD at F (§ 206).

$$\therefore \angle COF = \angle DOF. \qquad (?)$$

That is, OE bisects $\angle AOB$.

PROP. XXXII. PROBLEM.

208. *With a given vertex and a given side, to construct an angle equal to a given angle.*



Given O the vertex, and OA a side, of an $\angle$, and $\angle O'$.

Required to construct, with O as the vertex and OA as a side, an $\angle$ equal to $\angle O'$.

Construction. With O' as a centre, and any convenient radius, describe an arc intersecting the sides of $\angle O'$ at C and D; and draw chord CD.

With O as a centre, and with the same radius as before, describe the indefinite arc AE.

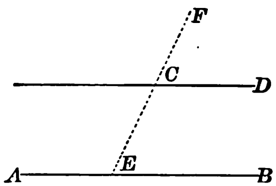
With A as a centre and CD as a radius, describe an arc intersecting arc AE at B, and draw line OB.

Then, $\angle AOB = \angle O'$.

(The chords of arcs AB and CD are equal, and the proposition follows by §§ 157 and 155.)

PROP. XXXIII. PROBLEM.

209. *Through a given point without a given straight line, to draw a parallel to the line.*



Given C any point without line AB.

Required to draw through C a line $\parallel$ to AB.

Construction. Through C draw any line EF, meeting AB at E, and construct $\angle FCD = \angle CEB$. (§ 208)

Then, $CD \parallel AB$. (?)

EXERCISES.

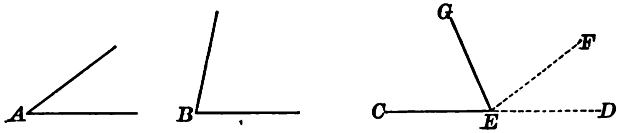
76. To construct an angle of 45°; of 135°; of $22\frac{1}{2}$°; of $67\frac{1}{2}$°.

77. Through a given point without a straight line to draw a line making a given angle with that line.

(Draw through the given point a $\parallel$ to the given line.)

PROP. XXXIV. PROBLEM.

210. *Given two angles of a triangle, to find the third.*



Given A and B two $\angle$s of a $\triangle$.

Required to construct the third $\angle$.

Construction. At any point E of the indefinite line CD, construct $\angle DEF = \angle A$. (§ 208)

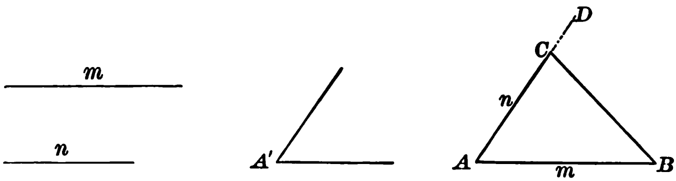
Also, $\angle FEG$ adjacent to $\angle DEF$, and equal to $\angle B$.

Then, $\angle CEG$ is the required $\angle$.

(The proof is left to the pupil.)

PROP. XXXV. PROBLEM.

211. *Given two sides and the included angle of a triangle, to construct the triangle.*



Given m and n two sides of a $\triangle$, and A' their included $\angle$.

Required to construct the $\triangle$.

Construction. Draw line $AB = m$.

Construct $\angle BAD = \angle A'$. (§ 208)

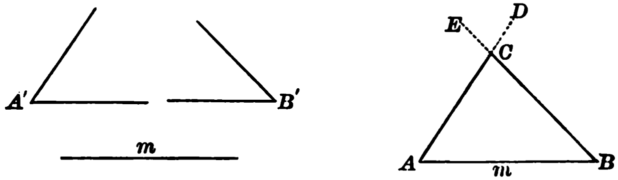
On AD take $AC = n$, and draw line BC.

Then, ABC is the required $\triangle$.

212. Sch. The problem is possible for any values of the given parts.

Prop. XXXVI. Problem.

213. *Given a side and two adjacent angles of a triangle, to construct the triangle.*



Given a side m, and the adj. $\angle$s A' and B' of a $\triangle$.

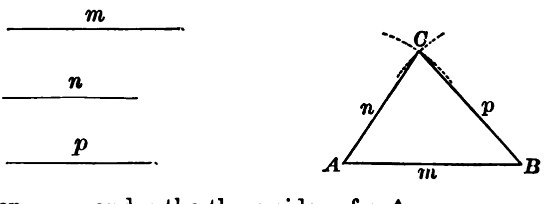
(The construction is left to the pupil.)

214. Sch. I. If a side and *any* two angles of a triangle are given, the third angle may be found by § 210, and the triangle may then be constructed as in § 213.

215. Sch. II. The problem is impossible when the sum of the given angles is equal to, or greater than, two right angles. (§ 84)

Prop. XXXVII. Problem.

216. *Given the three sides of a triangle, to construct the triangle.*



Given m, n, and p the three sides of a $\triangle$.

Required to construct the $\triangle$.

Construction. Draw line $AB = m$.

With A as a centre and n as a radius, describe an arc; with B as a centre and p as a radius, describe an arc intersecting the former arc at C, and draw lines AC and BC.

Then, ABC is the required $\triangle$.

217. Sch. The problem is impossible when one of the given sides is equal to, or greater than, the sum of the other two. (§ 61)

Prop. XXXVIII. Problem.

218. *Given two sides of a triangle, and the angle opposite to one of them, to construct the triangle.*

Given m and n two sides of a $\triangle$, and A' the $\angle$ opposite to n.

Required to construct the $\triangle$.

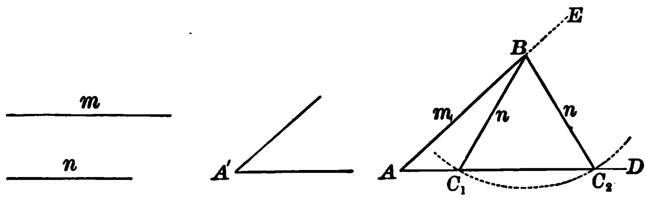
Construction. Construct $\angle DAE = \angle A'$ (§ 208), and on AE take $AB = m$.

With B as a centre and n as a radius, describe an arc.

Case I. *When A' is acute, and $m > n$.*

There may be three cases:

1. The arc may intersect AD in two points.



Let C_1 and C_2 be the points in which the arc intersects AD, and draw lines BC_1 and BC_2.

Then, either ABC_1 or ABC_2 is the required $\triangle$.

Note. This is called the *ambiguous case.*

2. The arc may be tangent to AD.

In this case there is but one $\triangle$.

And since a tangent to a $\odot$ is $\perp$ to the radius drawn to the point of contact (§ 170), the $\triangle$ is a *right* $\triangle$.

3. The arc may not intersect AD at all.

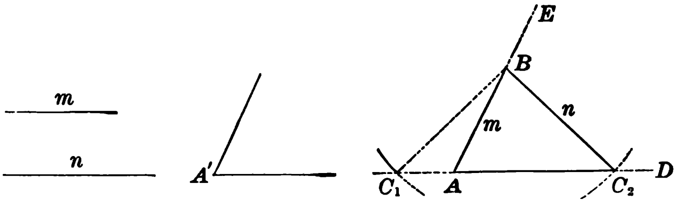
In this case the problem is impossible.

Case II. *When A' is acute, and $m = n$.*

In this case, the arc intersects AD in two points, one of which is A.

Then there is but one $\triangle$; an *isosceles* $\triangle$.

Case III. *When A' is acute, and $m < n$.*



In this case, the arc intersects AD in two points.

Let C_1 and C_2 be the points in which the arc intersects AD, and draw lines BC_1 and BC_2.

Now $\triangle ABC_1$ does not satisfy the conditions of the problem, since it does not contain the given $\angle A'$.

Then there is but one $\triangle$; $\triangle ABC_2$.

Case IV. *When A' is right or obtuse, and $m < n$.*

In each of these cases, the arc intersects AD in two points on opposite sides of A.

Then there is but one $\triangle$.

219. Sch. If A' is right or obtuse, and $m = n$ or $m > n$, the problem is impossible; for the side opposite the right or obtuse angle in a triangle must be the greatest side of the triangle. (§ 99)

The pupil should construct the triangle corresponding to each case of § 218.

EXERCISES.

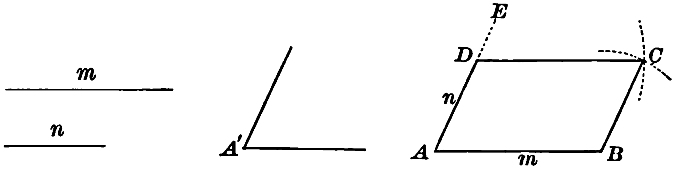
78. Given one of the equal sides and the altitude of an isosceles triangle, to construct the triangle.

What restriction is there on the values of the given lines?

79. Given two diagonals of a parallelogram and their included angle, to construct the parallelogram. (§ 111.)

PROP. XXXIX. PROBLEM.

220. *Given two sides and the included angle of a parallelogram, to construct the parallelogram.*

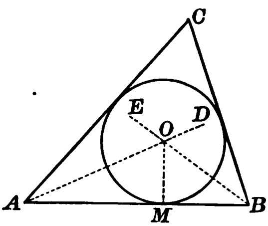


Given m and n two sides, and A' the included ∠, of a ▱.

(The construction and proof are left to the pupil.)

PROP. XL. PROBLEM.

221. *To inscribe a circle in a given triangle.*



Given △ ABC.

Required to inscribe a ⊙ in △ ABC.

Construction. Draw lines AD and BE bisecting ∠s A and B, respectively (§ 207).

From their intersection O, draw line $OM \perp AB$ (§ 204).

With O as a centre and OM as a radius, describe a ⊙.

This ⊙ will be tangent to AB, BC, and CA.

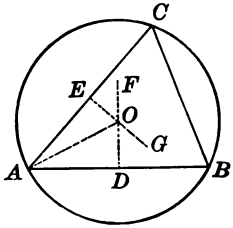
(The proof is left to the pupil; see § 135.)

Ex. 80. To construct a right triangle, having given the hypotenuse and an acute angle.

(The other acute ∠ is the complement of the given ∠.)

PROP. XLI. PROBLEM.

222. *To circumscribe a circle about a given triangle.*



Given $\triangle ABC$.

Required to circumscribe a $\odot$ about $\triangle ABC$.

Construction. Draw lines DF and EG $\perp$ to AB and AC, respectively, at their middle points (§ 205).

Let DF and EG intersect at O.

With O as a centre, and OA as a radius, describe a $\odot$.

The circumference will pass through A, B, and C.

(The proof is left to the pupil; see § 137.)

223. Sch. The above construction serves to describe a circumference through three given points not in the same straight line, or to find the centre of a given circumference or arc.

EXERCISES.

81. To construct a right triangle, having given a leg and the opposite acute angle.

(Construct the complement of the given $\angle$.)

82. Given the base and the vertical angle of an isosceles triangle, to construct the triangle.

(Each of the equal $\angle$s is the complement of one-half the vertical $\angle$.)

83. Given the altitude and one of the equal angles of an isosceles triangle, to construct the triangle.

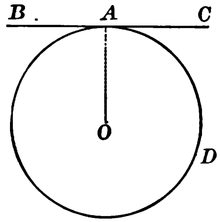
(One-half the vertical $\angle$ is the complement of each of the equal $\angle$s.)

84. To circumscribe a circle about a given rectangle.

(Draw the diagonals.)

Prop. XLII. Problem.

224. *To draw a tangent to a circle through a given point on the circumference.*



Given A any point on the circumference of ⊙ AD.

Required to draw through A a tangent to ⊙ AD.

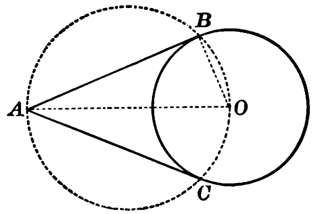
Construction. Draw radius OA.

Through A draw line $BC \perp OA$ (§ 203).

Then, BC will be tangent to ⊙ AD. (?)

Prop. XLIII. Problem.

225. *To draw a tangent to a circle through a given point without the circle.*



Given A any point without ⊙ BC.

Required to draw through A a tangent to ⊙ BC.

Construction. Let O be the centre of ⊙ BC, and draw line OA.

On OA as a diameter, describe a circumference, cutting the given circumference at B and C.

Draw lines AB and AC.

Then, AB and AC are tangents to $\odot BC$.

Proof. Draw line OB.

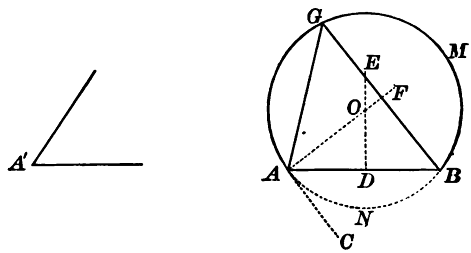
$\angle ABO$, being inscribed in a semicircle, is a rt. $\angle$. (?)

Therefore, AB is tangent to $\odot BC$. (?)

In like manner, AC is tangent to $\odot BC$.

PROP. XLIV. PROBLEM.

226. *Upon a given straight line, to describe a segment which shall contain a given angle.*



Given line AB, and $\angle A'$.

Required to describe upon AB a segment such that every $\angle$ inscribed in the segment shall equal $\angle A'$.

Construction. Construct $\angle BAC = \angle A'$. (§ 208)

Draw line $DE \perp$ to AB at its middle point. (§ 205)

Draw line $AF \perp AC$, intersecting DE at O.

With O as a centre and OA as a radius, describe $\odot AMBN$.

Then, AMB will be the required segment.

Proof. Let AGB be any $\angle$ inscribed in segment AMB.

Then, $\angle AGB$ is measured by $\frac{1}{2}$ arc ANB. (?)

But, by cons., $AC \perp OA$.

Whence, AC is tangent to $\odot AMB$. (?)

Therefore, $\angle BAC$ is measured by $\frac{1}{2}$ arc ANB. (§ 197)

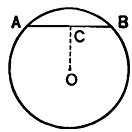
$$\therefore \angle AGB = \angle BAC = \angle A'. \quad (?)$$

Hence, every $\angle$ inscribed in segment AMB equals $\angle A'$. (§ 194)

EXERCISES.

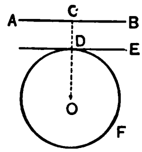
85. Given the middle point of a chord of a circle, to construct the chord.

(To draw through C a chord which is bisected at C.)



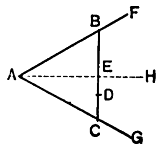
86. To draw a line tangent to a given circle and parallel to a given straight line.

(To draw a tangent ‖ AB.)



87. To draw a line tangent to a given circle and perpendicular to a given straight line.

88. To draw a straight line through a given point within a given acute ∠, forming with the sides of the angle an isosceles triangle.



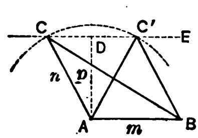
89. Given the base, an adjacent angle, and the altitude of a triangle, to construct the triangle.

(Draw a ‖ to the base at a distance equal to the altitude.)

90. Given the base, an adjacent side, and the altitude of a triangle, to construct the triangle.

Discuss the problem for the following cases:

1. $n > p$. 2. $n = p$. 3. $n < p$.



91. To construct a rhombus, having given its base and altitude.

(Draw a ‖ to the base at a distance equal to the altitude.)

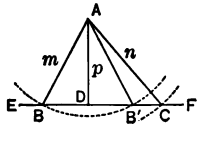
What restriction is there on the values of the given lines?

92. Given the altitude and the sides including the vertical angle of a triangle, to construct the triangle.

What restriction is there on the values of the given lines?

Discuss the problem for the following cases:

1. $m < n$ or $> n$. 2. $m = n$.



93. Given the altitude of a triangle, and the angles at the extremities of the base, to construct the triangle.

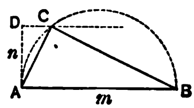
(The ∠ between the altitude and an adjacent side is the complement of the ∠ at the extremity of the base, if acute, or of its supplement, if obtuse.)

94. To construct an isosceles triangle, having given the base and the radius of the circumscribed circle.

What restriction is there on the values of the given lines?

95. To construct a square, having given one of its diagonals. (§ 195.)

96. To construct a right triangle, having given the hypotenuse and the length of the perpendicular drawn to it from the vertex of the right angle.

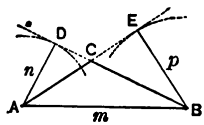


What restriction is there on the values of the given lines?

97. To construct a right triangle, having given the hypotenuse and a leg.

What restriction is there on the values of the given lines?

98. Given the base of a triangle and the perpendiculars from its extremities to the other sides, to construct the triangle. (§ 225.)



What restriction is there on the values of the given lines?

99. To describe a circle of given radius tangent to two given intersecting lines.

(Draw a ∥ to one of the lines at a distance equal to the radius.)

100. To describe a circle tangent to a given straight line, having its centre at a given point without the line.

101. To construct a circle having its centre in a given line, and passing through two given points without the line. (§ 163.)

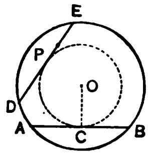
What restriction is there on the positions of the given points?

102. In a given straight line to find a point equally distant from two given intersecting lines. (§ 101.)

103. Given a side and the diagonals of a parallelogram, to construct the parallelogram.

What restriction is there on the values of the given lines?

104. Through a given point without a given straight line, to describe a circle tangent to the given line at a given point. (§ 163.)

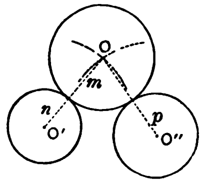


105. Through a given point within a circle to draw a chord equal to a given chord. (§ 164.)

What restriction is there on the position of the given point?

106. Through a given point to describe a circle of given radius tangent to a given straight line.

(Draw a ∥ to the given line at a distance equal to the radius.)



107. To describe a circle of given radius tangent to two given circles.

(To describe a ⊙ of radius m tangent to two given Ⓢ whose radii are n and p, respectively.)

What restriction is there on the value of m?

108. To describe a circle tangent to two given parallels, and passing through a given point.

What restriction is there on the position of the given point?

109. To describe a circle of given radius, tangent to a given line and a given circle.

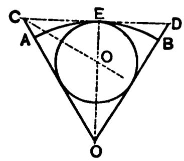
(Draw a ∥ to the given line at a distance equal to the given radius.)

110. To construct a parallelogram, having given a side, an angle, and the diagonal drawn from the vertex of the angle.

111. In a given triangle to inscribe a rhombus, having one of its angles coincident with an angle of the triangle.

(Bisect the ∠ which is common to the △ and the rhombus.)

112. To describe a circle touching two given intersecting lines, one of them at a given point. (§ 169.)



113. In a given sector to inscribe a circle.

(The problem is the same as inscribing a ⊙ in △ $O'CD$.)

114. In a given right triangle to inscribe a square, having one of its angles coincident with the right angle of the triangle.

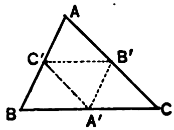
115. Through a vertex of a triangle to draw a straight line equally distant from the other vertices.

116. Given the base, the altitude, and the vertical angle of a triangle, to construct the triangle. (§ 226.)

(Construct on the given base as a chord a segment which shall contain the given ∠.)

117. Given the base of a triangle, its vertical angle, and the median drawn to the base, to construct the triangle.

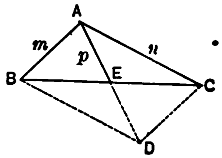
118. To construct a triangle, having given the middle points of its sides.



119. Given two sides of a triangle, and the median drawn to the third side, to construct the triangle.

(Construct $\triangle ABD$ with its sides equal to m, n, and $2p$, respectively.)

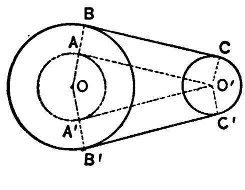
What restriction is there on the values of the given lines?

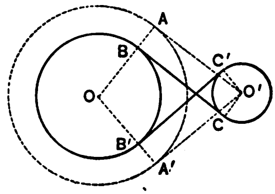


120. Given the base, the altitude, and the radius of the circumscribed circle of a triangle, to construct the triangle.

(The centre of the circumscribed ⊙ lies at a distance from each vertex equal to the radius of the ⊙.)

121. To draw common tangents to two given circles which do not intersect.





(To draw *exterior* common tangents, describe ⊙ AA' with its radius equal to the *difference* of the radii of the given ⊙s.

To draw *interior* common tangents, describe ⊙ AA' with its radius equal to the *sum* of the radii of the given ⊙s.)

Note. For additional exercises on Book II., see p. 224.

Book III.

THEORY OF PROPORTION. — SIMILAR POLYGONS.

DEFINITIONS.

227. A *Proportion* is a statement that two ratios are equal.

228. The statement that the ratio of a to b is equal to the ratio of c to d, may be written in either of the forms

$$a : b = c : d, \text{ or } \frac{a}{b} = \frac{c}{d}.$$

229. The first and fourth terms of a proportion are called the *extremes*, and the second and third terms the *means*.

The first and third terms are called the *antecedents*, and the second and fourth terms the *consequents*.

Thus, in the proportion $a : b = c : d$, a and d are the extremes, b and c the means, a and c the antecedents, and b and d the consequents.

230. If the means of a proportion are equal, either mean is called a *mean proportional* between the first and last terms, and the last term is called a *third proportional* to the first and second terms.

Thus, in the proportion $a : b = b : c$, b is a mean proportional between a and c, and c a third proportional to a and b.

231. A *fourth proportional* to three quantities is the fourth term of a proportion, whose first three terms are the three quantities taken in their order.

Thus, in the proportion $a : b = c : d$, d is a fourth proportional to a, b, and c.

PROP. I. THEOREM.

232. *In any proportion, the product of the extremes is equal to the product of the means.*

Given the proportion $a : b = c : d.$

To Prove $ad = bc.$

Proof. By § 228, $\frac{a}{b} = \frac{c}{d}.$

Multiplying both members of this equation by bd,

$$ad = bc.$$

233. Cor. *The mean proportional between two quantities is equal to the square root of their product.*

Given the proportion $a : b = b : c.$ (1)

To Prove $b = \sqrt{ac}.$

Proof. From (1), $b' = ac.$ (§ 232)

$$\therefore b = \sqrt{ac}.$$

PROP. II. THEOREM.

234. (Converse of Prop. I.) *If the product of two quantities is equal to the product of two others, one pair may be made the extremes, and the other pair the means, of a proportion.*

Given $ad = bc.$ (1)

To Prove $a : b = c : d.$

Proof. Dividing both members of (1) by bd,

$$\frac{ad}{bd} = \frac{bc}{bd}.$$

Or, $\frac{a}{b} = \frac{c}{d}.$

Then by § 228, $a : b = c : d.$

In like manner, $a : c = b : d,$

$b : a = d : c,$ etc.

PROP. III. THEOREM.

235. *In any proportion, the terms are in proportion by* ALTERNATION; *that is, the first term is to the third as the second term is to the fourth.*

Given the proportion $a : b = c : d$. (1)

To Prove $a : c = b : d$.

Proof. From (1), $ad = bc$. (§ 232)

$\therefore\ a : c = b : d$. (§ 234)

PROP. IV. THEOREM.

236. *In any proportion, the terms are in proportion by* INVERSION; *that is, the second term is to the first as the fourth term is to the third.*

Given the proportion $a : b = c : d$. (1)

To Prove $b : a = d : c$.

Proof. From (1), $ad = bc$. (?)

$\therefore\ b : a = d : c$. (?)

PROP. V. THEOREM.

237. *In any proportion, the terms are in proportion by* COMPOSITION; *that is, the sum of the first two terms is to the first term as the sum of the last two terms is to the third term.*

Given the proportion $a : b = c : d$. (1)

To Prove $a + b : a = c + d : c$.

Proof. From (1), $ad = bc$. (?)

Adding both members of the equation to ac,

$$ac + ad = ac + bc.$$

Factoring, $a(c + d) = c(a + b)$.

$\therefore\ a + b : a = c + d : c$. (§ 234)

In like manner, $a + b : b = c + d : d$.

PROP. VI. THEOREM.

238. *In any proportion, the terms are in proportion by* DIVISION; *that is, the difference of the first two terms is to the first term as the difference of the last two terms is to the third term.*

Given the proportion $a : b = c : d,$ (1)
in which $a > b$ and $c > d$.

To Prove $a - b : a = c - d : c.$

Proof. From (1), $ad = bc.$ (?)

Subtracting both members of the equation from ac,

$$ac - ad = ac - bc.$$

Factoring, $a(c - d) = c(a - b).$

$\therefore\ a - b : a = c - d : c.$ (?)

In like manner, $a - b : b = c - d : d.$

PROP. VII. THEOREM.

239. *In any proportion, the terms are in proportion by* COMPOSITION AND DIVISION; *that is, the sum of the first two terms is to their difference as the sum of the last two terms is to their difference.*

Given the proportion $a : b = c : d,$ (1)
in which $a > b$ and $c > d$.

To Prove $a + b : a - b = c + d : c - d.$

Proof. From (1), $\frac{a+b}{a} = \frac{c+d}{c},$ (§ 237)

and $\frac{a-b}{a} = \frac{c-d}{c}.$ (§ 238)

Dividing the first equation by the second,

$$\frac{a+b}{a-b} = \frac{c+d}{c-d}.$$

$$\therefore\ a + b : a - b = c + d : c - d.$$

Prop. VIII. Theorem.

240. *In a series of equal ratios, the sum of all the antecedents is to the sum of all the consequents as any antecedent is to its consequent.*

Given $a : b = c : d = e : f.$ (1)

To Prove $a + c + e : b + d + f = a : b.$

Proof. We have $ba = ab.$

Also, from (1), $bc = ad,$

and $be = af.$ (?)

Adding, $ba + bc + be = ab + ad + af.$

$\therefore\ b(a + c + e) = a(b + d + f).$

$\therefore\ a + c + e : b + d + f = a : b.$ (?)

Prop. IX. Theorem.

241. *In any proportion, like powers or like roots of the terms are in proportion.*

Given the proportion $a : b = c : d.$ (1)

To Prove $a^n : b^n = c^n : d^n.$

Proof. From (1), $\frac{a}{b} = \frac{c}{d}.$

Raising both members to the nth power,

$$\frac{a^n}{b^n} = \frac{c^n}{d^n}.$$

$$\therefore\ a^n : b^n = c^n : d^n.$$

In like manner, $\sqrt[n]{a} : \sqrt[n]{b} = \sqrt[n]{c} : \sqrt[n]{d}.$

Note. The ratio of two magnitudes of the same kind is equal to the ratio of their numerical measures when referred to a common unit (§ 183); hence, in any proportion involving the ratio of two magnitudes of the same kind, we may regard the ratio of the magnitudes as replaced by the ratio of their numerical measures when referred to a common unit.

Thus, let AB, CD, EF, and GH be four lines such that

$$AB : CD = EF : GH.$$

Then, $AB \times GH = CD \times EF.$ (§ 232)

This means that the product of the *numerical measures* of AB and GH is equal to the product of the *numerical measures* of CD and EF.

An interpretation of this nature must be given to all applications of §§ 232, 233 and 241.

EXERCISES.

1. Find a fourth proportional to 65, 80, and 91.

2. Find a mean proportional between 28 and 63.

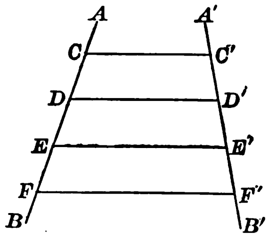
3. Find a third proportional to $\frac{3}{4}$ and $\frac{5}{6}$.

4. What is the second term of a proportion whose first, third, and fourth terms are 45, 160, and 224, respectively?

PROPORTIONAL LINES.

PROP. X. THEOREM.

242. *If a series of parallels, cutting two straight lines, intercept equal distances on one of these lines, they also intercept equal distances on the other.*



Given lines AB and $A'B'$ cut by ‖s CC', DD', EE', and FF' at points C, D, E, F, and C', D', E', F', respectively, so that $CD = DE = EF$.

To Prove $C'D' = D'E' = E'F'$.

Proof. In trapezoid $CC'E'E$, by hyp., DD' is ‖ to the bases, and bisects CE; it therefore bisects $C'E'$. (§ 133)

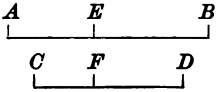
$$\therefore\ C'D' = D'E'. \qquad (1)$$

In like manner, in trapezoid $DD'F'F$, EE' is ‖ to the bases, and bisects DF.

$$\therefore\ D'E' = E'F'. \qquad (2)$$

From (1) and (2), $C'D' = D'E' = E'F'$. (?)

243. Def. Two straight lines are said to be *divided proportionally* when their corresponding segments are in the same ratio as the lines themselves.



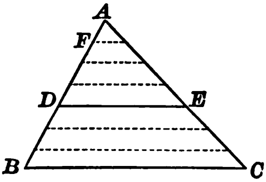
Thus, lines AB and CD are divided proportionally if

$$\frac{AE}{CF}=\frac{BE}{DF}=\frac{AB}{CD}.$$

PROP. XI. THEOREM.

244. *A parallel to one side of a triangle divides the other two sides proportionally.*

Case I. *When the segments of each side are commensurable.*



Given, in $\triangle ABC$, segments AD and BD of side AB commensurable, and line $DE \parallel BC$, meeting AC at E.

To Prove $$\frac{AD}{BD}=\frac{AE}{CE}.$$

Proof. Let AF be a common measure of AD and BD; and let it be contained 4 times in AD, and 3 times in BD.

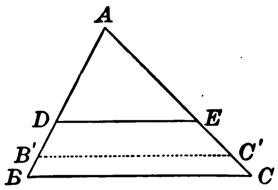
$$\therefore \frac{AD}{BD}=\frac{4}{3}. \qquad (1)$$

Drawing ∥s to BC through the several points of division of AB, AE will be divided into 4 parts, and CE into 3 parts, all of which parts are equal. (§ 242)

$$\therefore \frac{AE}{CE}=\frac{4}{3}. \qquad (2)$$

From (1) and (2), $$\frac{AD}{BD}=\frac{AE}{CE}. \qquad (?)$$

Case II. *When the segments of each side are incommensurable.*



Given, in $\triangle ABC$, segments AD and BD of side AB incommensurable, and line $DE \parallel BC$, meeting AC at E.

To Prove $\dfrac{AD}{BD} = \dfrac{AE}{CE}$.

Proof. Let AD be divided into any number of equal parts, and let one of these parts be applied to BD as a unit of measure.

Since AD and BD are incommensurable, a certain number of the equal parts will extend from D to B', leaving a remainder $BB' <$ one of the equal parts.

Draw line $B'C' \parallel BC$, meeting AC at C'.

Then, since AD and $B'D$ are commensurable,

$$\frac{AD}{B'D} = \frac{AE}{C'E}. \qquad \text{(§ 244, Case I.)}$$

Now let the number of subdivisions of AD be indefinitely increased.

Then the unit of measure will be indefinitely diminished, and the remainder BB' will approach the limit 0.

Then, $\dfrac{AD}{B'D}$ will approach the limit $\dfrac{AD}{BD}$,

and $\dfrac{AE}{C'E}$ will approach the limit $\dfrac{AE}{CE}$.

By the Theorem of Limits, these limits are equal. (?)

$$\therefore \frac{AD}{BD} = \frac{AE}{CE}.$$

245. Cor. I. We may write the result of § 244,

$$AD : BD = AE : CE. \quad (1)$$

$$\therefore AD + BD : AD = AE + CE : AE. \quad (\S\ 237)$$

$$\therefore AB : AD = AC : AE. \quad (2)$$

In like manner, $AB : BD = AC : CE. \quad (3)$

246. Cor. II. From (2), § 245,

$$AB : AC = AD : AE,$$

and from (3), $AB : AC = BD : CE. \quad (\S\ 235)$

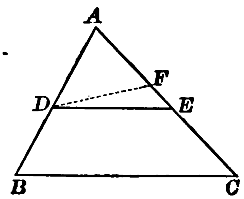
Then, by Ax. 1, $\frac{AB}{AC} = \frac{AD}{AE} = \frac{BD}{CE}. \quad (4)$

247. Sch. The proportions (1), (2), (3), and (4), of §§ 245 and 246, are all included in the general statement,

A parallel to one side of a triangle divides the other two sides proportionally.

PROP. XII. THEOREM.

248. (Converse of Prop. XI.) *A line which divides two sides of a triangle proportionally is parallel to the third side.*



Given, in $\triangle ABC$, line DE meeting AB and AC at D and E respectively, so that

$$\frac{AB}{AD} = \frac{AC}{AE}.$$

To Prove $DE \parallel BC$.

Proof. If DE is not $\parallel BC$, draw line $DF \parallel BC$, meeting AC at F.

$$\therefore \frac{AB}{AD} = \frac{AC}{AF}. \quad (\S\ 247)$$

But by hyp., $$\frac{AB}{AD} = \frac{AC}{AE}.$$

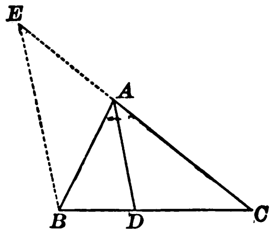
$$\therefore \frac{AC}{AE} = \frac{AC}{AF}. \qquad (?)$$

$$\therefore AE = AF.$$

Then, DF coincides with DE, and $DE \parallel BC$. (Ax. 3)

Prop. XIII. Theorem.

249. *In any triangle, the bisector of an angle divides the opposite side into segments proportional to the adjacent sides.*



Given line AD bisecting $\angle A$ of $\triangle ABC$, meeting BC at D.

To Prove $$\frac{DB}{DC} = \frac{AB}{AC}.$$

Proof. Draw line $BE \parallel AD$, meeting CA produced at E.

Then, since ||s AD and BE are cut by AB,

$$\angle ABE = \angle BAD. \qquad (?)$$

And since ||s AD and BE are cut by CE,

$$\angle AEB = \angle CAD. \qquad (?)$$

But by hyp., $$\angle BAD = \angle CAD.$$

$$\therefore \angle ABE = \angle AEB. \qquad (?)$$

$$\therefore AB = AE. \qquad (?)$$

Now since AD is || to side BE of $\triangle BCE$,

$$\frac{DB}{DC} = \frac{AE}{AC}. \qquad (\S\ 247)$$

Putting for AE its equal AB, we have

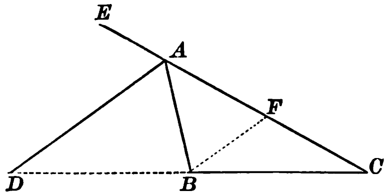
$$\frac{DB}{DC} = \frac{AB}{AC}.$$

250. Def. The *segments* of a line by a point are the distances from the point to the extremities of the line, whether the point is in the line itself, or in the line produced.

PROP. XIV. THEOREM.

251. *In any triangle the bisector of an exterior angle divides the opposite side externally into segments proportional to the adjacent sides.*

Note. The theorem does not hold for the exterior angle at the vertex of an *isosceles* triangle.



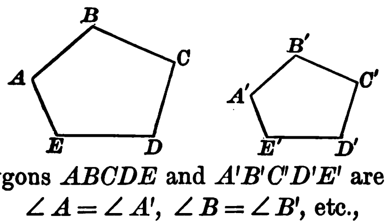
Given line AD bisecting ext. $\angle BAE$ of $\triangle ABC$, meeting CB produced at D.

To Prove $$\frac{DB}{DC}=\frac{AB}{AC}.$$

(Draw $BF \parallel AD$; then $\angle ABF = \angle AFB$, and $AF = AB$; BF is $\parallel$ to side AD of $\triangle ACD$.)

SIMILAR POLYGONS.

252. Def. Two polygons are said to be *similar* if they are mutually equiangular (§ 122), and have their homologous sides (§ 123) proportional.



Thus, polygons $ABCDE$ and $A'B'C'D'E'$ are similar if
$$\angle A = \angle A',\ \angle B = \angle B',\ \text{etc.},$$
and, $$\frac{AB}{A'B'}=\frac{BC}{B'C'}=\frac{CD}{C'D'},\ \text{etc.}$$

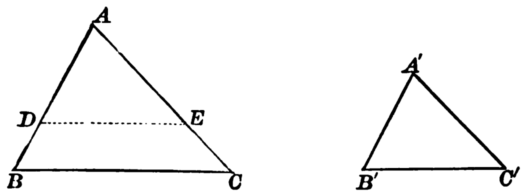
253. Sch. The following are given for reference:

1. *In similar polygons, the homologous angles are equal.*

2. *In similar polygons, the homologous sides are proportional.*

PROP. XV. THEOREM.

254. *Two triangles are similar when they are mutually equiangular.*



Given, in $\triangle$ ABC and $A'B'C'$,

$$\angle A = \angle A', \ \angle B = \angle B', \text{ and } \angle C = \angle C'.$$

To Prove $\triangle$ ABC and $A'B'C'$ similar.

Proof. Place $\triangle A'B'C'$ in the position ADE; $\angle A'$ coinciding with its equal $\angle A$, vertices B' and C' falling at D and E, respectively, and side $B'C'$ at DE.

Since, by hyp., $\angle ADE = \angle B$, $DE \parallel BC$. (?)

$$\therefore \frac{AB}{AD} = \frac{AC}{AE}. \qquad (\S\ 247)$$

That is,
$$\frac{AB}{A'B'} = \frac{AC}{A'C'}. \qquad (1)$$

In like manner, by placing $\triangle A'B'C'$ so that $\angle B'$ shall coincide with its equal $\angle B$, vertices A' and C' falling on sides AB and BC, respectively, we may prove

$$\frac{AB}{A'B'} = \frac{BC}{B'C'}. \qquad (2)$$

From (1) and (2),
$$\frac{AB}{A'B'} = \frac{AC}{A'C'} = \frac{BC}{B'C'}. \qquad (?)$$

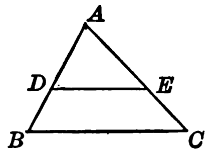
Then, $\triangle$ ABC and $A'B'C'$ have their homologous sides proportional, and are similar. (§ 252)

255. Cor. I. *Two triangles are similar when two angles of one are equal respectively to two angles of the other.*

For their remaining ∠s are equal each to each. (§ 86)

256. Cor. II. *Two right triangles are similar when an acute angle of one is equal to an acute angle of the other.*

257. Cor. III. *If a line be drawn between two sides of a triangle parallel to the third side, the triangle formed is similar to the given triangle.*



Given line $DE \parallel$ to side BC of $\triangle ABC$, meeting AB and AC at D and E, respectively.

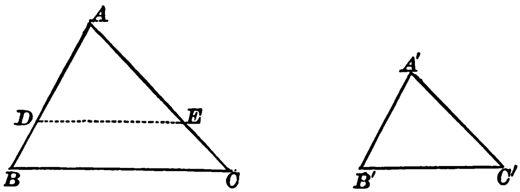
To Prove $\triangle ADE$ similar to $\triangle ABC$.

(The △s are mutually equiangular.)

258. Sch. *In similar triangles, the homologous sides lie opposite the equal angles.*

PROP. XVI. THEOREM.

259. *Two triangles are similar when their homologous sides are proportional.*



Given, in △s ABC and $A'B'C'$,

$$\frac{AB}{A'B'} = \frac{AC}{A'C'} = \frac{BC}{B'C'}.$$

To Prove △s ABC and $A'B'C'$ similar.

Proof. On AB and AC, take $AD = A'B'$ and $AE = A'C'$. Draw line DE; then, from the given proportion,

$$\frac{AB}{AD} = \frac{AC}{AE}.$$

$\therefore\ DE \parallel BC.$ (§ 248)

Then, $\triangle$s ADE and ABC are similar. (§ 257)

$$\therefore\ \frac{AB}{AD}=\frac{BC}{DE}, \text{ or } \frac{AB}{A'B'}=\frac{BC}{DE}. \qquad (§\ 253,\ 2)$$

But by hyp., $$\frac{AB}{A'B'}=\frac{BC}{B'C'}.$$

$$\therefore\ DE = B'C'.$$

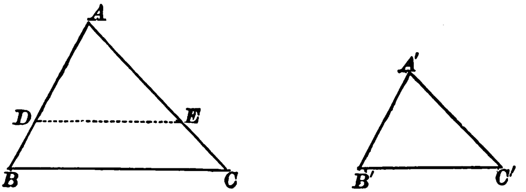
$$\therefore\ \triangle ADE = \triangle A'B'C'. \qquad (§\ 69)$$

But, $\triangle ADE$ has been proved similar to $\triangle ABC$.
Hence, $\triangle A'B'C'$ is similar to $\triangle ABC$.

260. Sch. To prove that two polygons in general are similar, it must be shown that they are mutually equiangular, and have their homologous sides proportional (§ 252); but in the case of two *triangles*, each of these conditions involves the other (§§ 254, 259), so that it is only necessary to show that one of the tests of similarity is satisfied.

Prop. XVII. Theorem.

261. *Two triangles are similar when they have an angle of one equal to an angle of the other, and the sides including these angles proportional.*



Given, in $\triangle$s ABC and $A'B'C'$,

$$\angle A = \angle A', \text{ and } \frac{AB}{A'B'}=\frac{AC}{A'C'}.$$

To Prove $\triangle$s ABC and $A'B'C'$ similar.

(Place $\triangle A'B'C'$ in the position ADE; by § 248, $DE \parallel BC$; the theorem follows by § 257.)

Prop. XVIII. Theorem.

262. *Two triangles are similar when their sides are parallel each to each, or perpendicular each to each.*

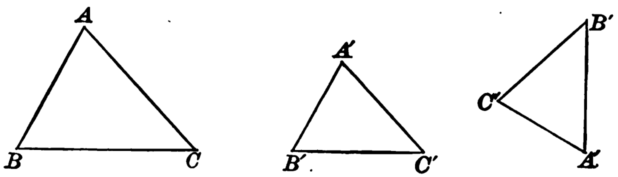


Fig. 1. *Fig. 2.* *Fig. 3.*

Given sides AB, AC, and BC, of $\triangle ABC$, ∥ respectively to sides $A'B'$, $A'C'$, and $B'C'$ of $\triangle A'B'C'$ in Fig. 2, and ⊥ respectively to sides $A'B'$, $A'C'$, and $B'C'$ of $\triangle A'B'C'$ in Fig. 3.

To Prove △ ABC and $A'B'C'$ similar.

Proof. Since the sides of ∠s A and A' are ∥ each to each, or ⊥ each to each, ∠s A and A' are either equal or supplementary. (§§ 81, 82, 83)

In like manner, ∠s B and B', and ∠s C and C', are either equal or supplementary.

We may then make the following hypotheses with regard to the ∠s of the △s:

1. $A + A' = 2$ rt. ∠s, $B + B' = 2$ rt. ∠s, $C + C' = 2$ rt. ∠s.
2. $A + A' = 2$ rt. ∠s, $B + B' = 2$ rt. ∠s, $C = C'$.
3. $A + A' = 2$ rt. ∠s, $B = B'$, $C + C' = 2$ rt. ∠s.
4. $A = A'$, $B + B' = 2$ rt. ∠s, $C + C' = 2$ rt. ∠s.
5. $A = A'$, $B = B'$, whence $C = C'$. (§ 86)

The first four hypotheses are impossible; for, in either case, the sum of the six ∠s of the two △s would be $>$ 4 rt. ∠s. (§ 84)

We can then have only $A = A'$, $B = B'$, and $C = C'$.

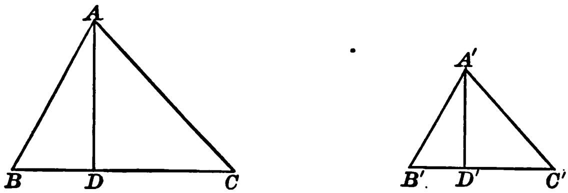
Therefore, △ ABC and $A'B'C'$ are similar. (§ 254)

263. Sch. 1. *In similar triangles whose sides are parallel each to each, the parallel sides are homologous.*

2. *In similar triangles whose sides are perpendicular each to each, the perpendicular sides are homologous.*

Prop. XIX. Theorem.

264. *The homologous altitudes of two similar triangles are in the same ratio as any two homologous sides.*



Given AD and $A'D'$ homologous altitudes of similar $\triangle ABC$ and $A'B'C'$.

To Prove $\frac{AD}{A'D'}=\frac{AB}{A'B'}=\frac{AC}{A'C'}=\frac{BC}{B'C'}$.

(Rt. $\triangle ABD$ and $A'B'D'$ are similar by § 256.)

265. Sch. In two similar triangles, any two homologous lines are in the same ratio as any two homologous sides.

EXERCISES.

5. The sides of a triangle are $AB = 8$, $BC = 6$, and $CA = 7$; find the segments into which each side is divided by the bisector of the opposite angle.

6. The sides of a triangle are $AB = 5$, $BC = 7$, and $CA = 8$; find the segments into which each side is divided by the bisector of the exterior angle at the opposite vertex.

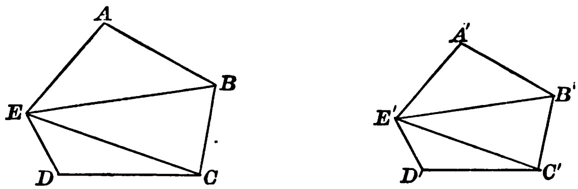
7. The sides of a triangle are 5, 7, and 9. The shortest side of a similar triangle is 14. What are the other two sides?

8. Two isosceles triangles are similar when their vertical angles are equal. (§ 255.)

9. The base and altitude of a triangle are 5 ft. 10 in. and 3 ft. 6 in., respectively. If the homologous base of a similar triangle is 7 ft. 6 in., find its homologous altitude.

Prop. XX. Theorem.

266. *Two polygons are similar when they are composed of the same number of triangles, similar each to each, and similarly placed.*



Given, in polygons AC and $A'C'$, $\triangle ABE$ similar to $\triangle A'B'E'$, $\triangle BCE$ to $\triangle B'C'E'$, and $\triangle CDE$ to $\triangle C'D'E'$.

To Prove polygons AC and $A'C'$ similar.

Proof. Since $\triangle$s ABE and $A'B'E'$ are similar,

$$\angle A = \angle A'. \qquad (?)$$

Also, $$\angle ABE = \angle A'B'E'.$$

And since $\triangle$s BCE and $B'C'E'$ are similar,

$$\angle EBC = \angle E'B'C'.$$

$$\therefore \angle ABE + \angle EBC = \angle A'B'E' + \angle E'B'C'.$$

Or, $$\angle ABC = \angle A'B'C'.$$

In like manner, $\angle BCD = \angle B'C'D'$, etc.

Then, AC and $A'C'$ are mutually equiangular.

Again, since $\triangle ABE$ is similar to $\triangle A'B'E'$, and $\triangle BCE$ to $\triangle B'C'E'$,

$$\frac{AB}{A'B'} = \frac{BE}{B'E'} \text{ and } \frac{BE}{B'E'} = \frac{BC}{B'C'}. \qquad (?)$$

$$\therefore \frac{AB}{A'B'} = \frac{BC}{B'C'}. \qquad (?)$$

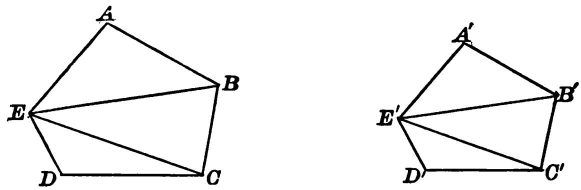
In like manner, $$\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CD}{C'D'}, \text{ etc.}$$

Then, AC and $A'C'$ have their homologous sides proportional.

Therefore, AC and $A'C'$ are similar. (§ 252)

PROP. XXI. THEOREM.

267. (Converse of Prop. XX.) *Two similar polygons may be decomposed into the same number of triangles, similar each to each, and similarly placed.*



Given E and E' homologous vertices of similar polygons AC and $A'C'$, and lines EB, EC, $E'B'$, and $E'C'$.

To Prove $\triangle ABE$ similar to $\triangle A'B'E'$, $\triangle BCE$ to $\triangle B'C'E'$, and $\triangle CDE$ to $\triangle C'D'E'$.

Proof. Since polygons AC and $A'C'$ are similar,

$$\angle A = \angle A' \text{ and } \frac{AE}{A'E'} = \frac{AB}{A'B'}. \qquad (?)$$

Then, $\triangle ABE$ and $A'B'E'$ are similar. (§ 261)

Again, since the polygons are similar,

$$\angle ABC = \angle A'B'C'.$$

And since $\triangle ABE$ and $A'B'E'$ are similar,

$$\angle ABE = \angle A'B'E'.$$

$$\therefore \angle ABC - \angle ABE = \angle A'B'C' - \angle A'B'E'.$$

Or, $$\angle EBC = \angle E'B'C'.$$

Also, since the polygons are similar, $\dfrac{AB}{A'B'} = \dfrac{BC}{B'C'}$.

And since $\triangle ABE$ and $A'B'E'$ are similar, $\dfrac{AB}{A'B'} = \dfrac{BE}{B'E'}$.

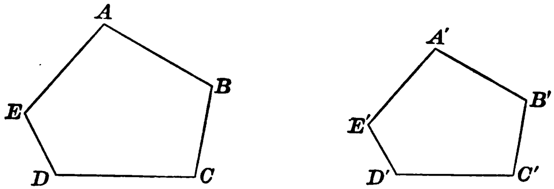
$$\therefore \frac{BC}{B'C'} = \frac{BE}{B'E'}. \qquad (?)$$

Then, since $\angle EBC = \angle E'B'C'$, and $\dfrac{BC}{B'C'} = \dfrac{BE}{B'E'}$, $\triangle BCE$ and $B'C'E'$ are similar. (?)

In like manner, we may prove $\triangle CDE$ and $C'D'E'$ similar.

PROP. XXII. THEOREM.

268. *The perimeters of two similar polygons are in the same ratio as any two homologous sides.*



Given AB and $A'B'$, BC and $B'C'$, CD and $C'D'$, etc., homologous sides of similar polygons AC and $A'C'$.

To Prove

$$\frac{AB + BC + CD + \text{etc.}}{A'B' + B'C' + C'D' + \text{etc.}} = \frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CD}{C'D'}, \text{ etc.}$$

(Apply § 240 to the equal ratios of § 252.)

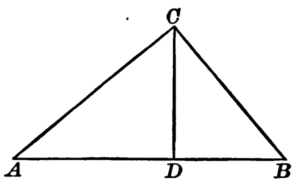
PROP. XXIII. THEOREM.

269. *If a perpendicular be drawn from the vertex of the right angle to the hypotenuse of a right triangle,*

I. *The triangles formed are similar to the whole triangle, and to each other.*

II. *The perpendicular is a mean proportional between the segments of the hypotenuse.*

III. *Either leg is a mean proportional between the whole hypotenuse and the adjacent segment.*



Given line $CD \perp$ hypotenuse AB of rt. $\triangle ABC$.

I. **To Prove** △s *ACD* and *BCD* similar to △ *ABC*, and to each other.

Proof. In rt. △s *ACD* and *ABC*,

$$\angle A = \angle A.$$

Then, △ *ACD* is similar to △ *ABC*. (§ 256)

In like manner, △ *BCD* is similar to △ *ABC*.

Then, △s *ACD* and *BCD* are similar to each other, for each is similar to △ *ABC*.

II. **To Prove** $AD : CD = CD : BD.$

Proof. Since △s *ACD* and *BCD* are similar,

$$\angle ACD = \angle B \text{ and } \angle A = \angle BCD. \qquad (\S\ 253,\ 1)$$

In △s *ACD* and *BCD*, *AD* and *CD* are homologous sides, for they lie opposite the equal ∠s *ACD* and *B*, respectively; also, *CD* and *BD* are homologous sides, for they lie opposite the equal ∠s *A* and *BCD*, respectively. (§ 258)

$$\therefore AD : CD = CD : BD. \qquad (?)$$

III. **To Prove** $AB : AC = AC : AD.$

Proof. Since △s *ABC* and *ACD* are similar,

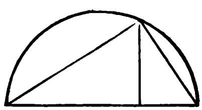
$$\angle ACB = \angle ADC \text{ and } \angle B = \angle ACD. \qquad (?)$$

In △s *ABC* and *ACD*, *AB* and *AC* are homologous sides, for they lie opposite the equal ∠s *ACB* and *ADC*, respectively; also, *AC* and *AD* are homologous sides, for they lie opposite the equal ∠s *B* and *ACD*, respectively. (?)

$$\therefore AB : AC = AC : AD. \qquad (?)$$

In like manner, $AB : BC = BC : BD.$

270. Cor. I. Since an angle inscribed in a semicircle is a right angle (§ 195), it follows that:



If a perpendicular be drawn from any point in the circumference of a circle to a diameter,

1. *The perpendicular is a mean proportional between the segments of the diameter.*

2. *The chord joining the point to either extremity of the diameter is a mean proportional between the whole diameter and the adjacent segment.*

271. Cor. II. The three proportions of § 269 give

$$\overline{CD}^2 = AD \times BD,$$

$$\overline{AC}^2 = AB \times AD,$$

and $$\overline{BC}^2 = AB \times BD. \quad (?)$$

Hence, *if a perpendicular be drawn from the vertex of the right angle to the hypotenuse of a right triangle,*

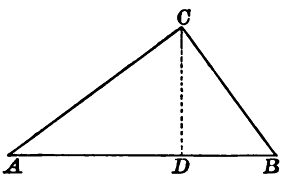
1. *The square of the perpendicular is equal to the product of the segments of the hypotenuse.*

2. *The square of either leg is equal to the product of the whole hypotenuse and the adjacent segment.*

As stated in Note, p. 126, these equations mean that the square of the *numerical measure* of CD is equal to the product of the *numerical measures* of AD and BD, etc.

Prop. XXIV. Theorem.

272. *In any right triangle, the square of the hypotenuse is equal to the sum of the squares of the legs.*



Given AB the hypotenuse of rt. $\triangle ABC$.

To Prove $\overline{AB}^2 = \overline{AC}^2 + \overline{BC}^2$.

Proof. Draw line $CD \perp AB$.

Then, $$\overline{AC}^2 = AB \times AD,$$

and $$\overline{BC}^2 = AB \times BD. \quad (§ 271, 2)$$

Adding, $$\overline{AC}^2 + \overline{BC}^2 = AB \times (AD + BD) = AB \times AB.$$

$$\therefore \overline{AB}^2 = \overline{AC}^2 + \overline{BC}^2.$$

273. Cor. I. It follows from § 272 that

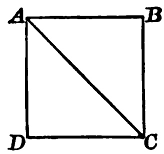
$$\overline{AC}^2 = \overline{AB}^2 - \overline{BC}^2, \text{ and } \overline{BC}^2 = \overline{AB}^2 - \overline{AC}^2.$$

That is, *in any right triangle, the square of either leg is equal to the square of the hypotenuse, minus the square of the other leg.*

274. Cor. II. If AC is a diagonal of square $ABCD$,

$$\overline{AC}^2 = \overline{AB}^2 + \overline{BC}^2 = \overline{AB}^2 + \overline{AB}^2. \quad (\S\ 272)$$

$$\therefore\ \overline{AC}^2 = 2\,\overline{AB}^2.$$



Dividing both members by $\overline{AB}^2$,

$$\frac{\overline{AC}^2}{\overline{AB}^2} = 2, \text{ or } \frac{AC}{AB} = \sqrt{2}.$$

Hence, *the diagonal of a square is incommensurable with its side* (§ 181).

EXERCISES.

10. The perimeters of two similar polygons are 119 and 68; if a side of the first is 21, what is the homologous side of the second?

11. What is the length of the tangent to a circle whose diameter is 16, from a point whose distance from the centre is 17?

12. What is the length of the longest straight line which can be drawn on a floor 33 ft. 4 in. long, and 16 ft. 3 in. wide?

13. A ladder 32 ft. 6 in. long is placed so that it just reaches a window 26 ft. above the street; and when turned about its foot, just reaches a window 16 ft. 6 in. above the street on the other side. Find the width of the street.

14. The altitude of an equilateral triangle is 5; what is its side?

15. Find the length of the diagonal of a square whose side is 1 ft. 3 in.

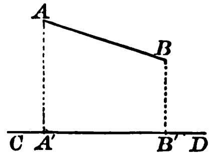
16. One of the non-parallel sides of a trapezoid is perpendicular to the bases. If the length of this side is 40, and of the parallel sides 31 and 22, respectively, what is the length of the other side?

17. The length of the tangent to a circle, whose diameter is 80, from a point without the circumference, is 42. What is the distance of the point from the centre?

18. If the length of the common chord of two intersecting circles is 16, and their radii are 10 and 17, what is the distance between their centres? (§ 178.)

DEFINITIONS.

275. The *projection of a point* upon a straight line of indefinite length, is the foot of the perpendicular drawn from the point to the line.



Thus, if line AA' be perpendicular to line CD, the projection of point A on line CD is point A'.

276. The *projection of a finite straight line* upon a straight line of indefinite length, is that portion of the second line included between the projections of the extremities of the first.

Thus, if lines AA' and BB' be perpendicular to line CD, the projection of line AB upon line CD is line $A'B'$.

PROP. XXV. THEOREM

277. *In any triangle, the square of the side opposite an acute angle is equal to the sum of the squares of the other two sides, minus twice the product of one of these sides and the projection of the other side upon it.*

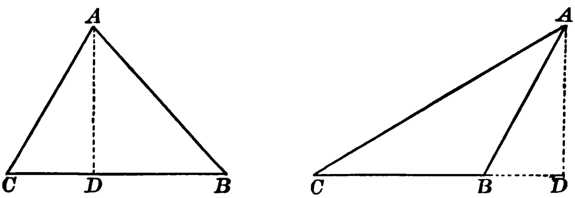


Fig. 1. *Fig. 2.*

Given C an acute $\angle$ of $\triangle ABC$, and CD the projection of side AC upon side CB, produced if necessary. (§ 276)

To Prove $\overline{AB}^2 = \overline{BC}^2 + \overline{AC}^2 - 2\,BC \times CD$.

Proof. Draw line AD; then, $AD \perp CD$. (§ 276)

There will be two cases according as D falls on CB (Fig. 1), or on CB produced (Fig. 2).

In Fig. 1, $BD = BC - CD.$

In Fig. 2, $BD = CD - BC.$

Squaring both members of the equation, we have by the algebraic rule for the square of the difference of two numbers, in either case,

$$\overline{BD}^2 = \overline{BC}^2 + \overline{CD}^2 - 2\,BC \times CD.$$

Adding $\overline{AD}^2$ to both members,

$$\overline{AD}^2 + \overline{BD}^2 = \overline{BC}^2 + \overline{AD}^2 + \overline{CD}^2 - 2\,BC \times CD.$$

But in rt. ⊿ ABD and ACD,

$$\overline{AD}^2 + \overline{BD}^2 = \overline{AB}^2,$$

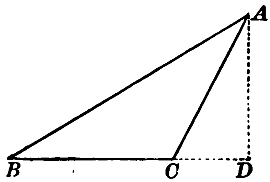
and $$\overline{AD}^2 + \overline{CD}^2 = \overline{AC}^2.$$ (§ 272)

Substituting these values, we have

$$\overline{AB}^2 = \overline{BC}^2 + \overline{AC}^2 - 2\,BC \times CD.$$

PROP. XXVI. THEOREM.

278. *In any triangle having an obtuse angle, the square of the side opposite the obtuse angle is equal to the sum of the squares of the other two sides, plus twice the product of one of these sides and the projection of the other side upon it.*



Given C an obtuse ∠ of △ ABC, and CD the projection of side AC upon side BC produced.

To Prove $\overline{AB}^2 = \overline{BC}^2 + \overline{AC}^2 + 2\,BC \times CD.$

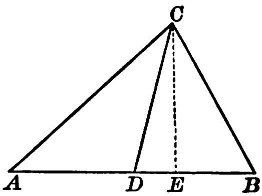
(We have $BD = BC + CD$; square both members, using the algebraic rule for the square of the sum of two numbers, and then add $\overline{AD}^2$ to both members.)

PROP. XXVII. THEOREM.

279. *In any triangle, if a median be drawn from the vertex to the base,*

I. *The sum of the squares of the other two sides is equal to twice the square of half the base, plus twice the square of the median.*

II. *The difference of the squares of the other two sides is equal to twice the product of the base and the projection of the median upon the base.*



Given DE the projection of median CD upon base AB of $\triangle ABC$; and $AC > BC$.

To Prove I. $\overline{AC}^2 + \overline{BC}^2 = 2\,\overline{AD}^2 + 2\,\overline{CD}^2$.

II. $\overline{AC}^2 - \overline{BC}^2 = 2\,AB \times DE$.

Proof. Since $AC > BC$, E falls between B and D.

Then, $\angle ADC$ is obtuse, and $\angle BDC$ acute.

Hence, in $\triangle$s ADC and BDC,

$$\overline{AC}^2 = \overline{AD}^2 + \overline{CD}^2 + 2\,AD \times DE, \qquad (\S\ 278)$$

and
$$\overline{BC}^2 = \overline{BD}^2 + \overline{CD}^2 - 2\,BD \times DE. \qquad (\S\ 277)$$

But by hyp., $BD = AD$.

$$\therefore \overline{AC}^2 = \overline{AD}^2 + \overline{CD}^2 + AB \times DE, \qquad (1)$$

and
$$\overline{BC}^2 = \overline{AD}^2 + \overline{CD}^2 - AB \times DE. \qquad (2)$$

Adding (1) and (2), we have

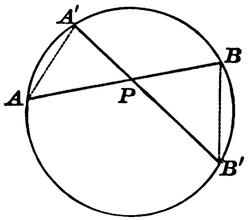
$$\overline{AC}^2 + \overline{BC}^2 = 2\,\overline{AD}^2 + 2\,\overline{CD}^2.$$

Subtracting (2) from (1), we have

$$\overline{AC}^2 - \overline{BC}^2 = 2\,AB \times DE.$$

PROP. XXVIII. THEOREM.

280. *If any two chords be drawn through a fixed point within a circle, the product of the segments of one chord is equal to the product of the segments of the other.*



Given AB and $A'B'$ any two chords passing through fixed point P within $\odot AA'B$.

To Prove $AP \times BP = A'P \times B'P$.

Proof. Draw lines AA' and BB'.

Then, in $\triangle AA'P$ and $BB'P$,

$$\angle A = \angle B',$$

for each is measured by $\frac{1}{2}$ arc $A'B$. (?)

In like manner, $\angle A' = \angle B$.

Then, $\triangle AA'P$ and $BB'P$ are similar. (?)

In similar $\triangle AA'P$ and $BB'P$, sides AP and $B'P$ are homologous, as also are sides $A'P$ and BP. (§ 258)

$$\therefore AP : A'P = B'P : BP. \quad (?)$$

$$\therefore AP \times BP = A'P \times B'P. \quad (?)$$

281. Sch. The proportion of § 280 may be written

$$\frac{AP}{A'P} = \frac{B'P}{BP}, \text{ or } \frac{AP}{A'P} = \frac{1}{\frac{BP}{B'P}}.$$

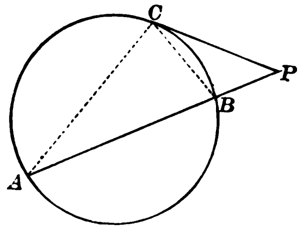
If two magnitudes, such as the segments of a chord passing through a fixed point, are so related that the ratio of any two values of one is equal to the *reciprocal* of the ratio of the corresponding values of the other, they are said to be *reciprocally proportional.*

Then, the theorem may be written,

If any two chords be drawn through a fixed point within a circle, their segments are reciprocally proportional.

PROP. XXIX. THEOREM.

282. *If through a fixed point without a circle a secant and a tangent be drawn, the product of the whole secant and its external segment is equal to the square of the tangent.*



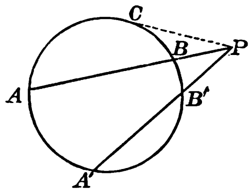
Given AP a secant, and CP a tangent, passing through fixed point P without $\odot ABC$.

To Prove $AP \times BP = \overline{CP}^2$.

($\angle A = \angle BCP$, for each is measured by $\frac{1}{2}$ arc BC (?); then $\triangle ACP$ and BCP are similar, and their homologous sides are proportional.)

283. Cor. I. *If through a fixed point without a circle a secant and a tangent be drawn, the tangent is a mean proportional between the whole secant and its external segment.*

284. Cor. II. *If any two secants be drawn through a fixed point without a circle, the product of one and its external segment is equal to the product of the other and its external segment.*



Given P any point without $\odot ABC$, and AP and $A'P$ secants intersecting the circumference at A and B, and A' and B', respectively.

To Prove $AP \times BP = A'P \times B'P$.

(We have $AP \times BP$ and $A'P \times B'P$ each equal to $\overline{CP}^2$.)

285. Cor. III. *If any two secants be drawn through a fixed point without a circle, the whole secants and their external segments are reciprocally proportional* (§ 281).

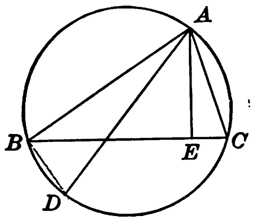
EXERCISES.

19. Find the length of the common tangent to two circles whose radii are 11 and 18, if the distance between their centres is 25.

20. AB is the hypotenuse of right triangle ABC. If perpendiculars be drawn to AB at A and B, meeting AC produced at D, and BC produced at E, prove triangles ACE and BCD similar.

Prop. XXX. Theorem.

286. *In any triangle, the product of any two sides is equal to the diameter of the circumscribed circle, multiplied by the perpendicular drawn to the third side from the vertex of the opposite angle.*



Given AD a diameter of the circumscribed $\odot ACD$ of $\triangle ABC$, and line $AE \perp BC$.

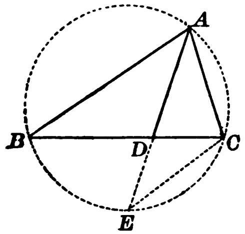
To Prove $AB \times AC = AD \times AE$.

(In rt. $\triangle$ ABD and ACE, $\angle D = \angle C$; then, the $\triangle$ are similar, and their homologous sides are proportional.)

287. Cor. *In any triangle, the diameter of the circumscribed circle is equal to the product of any two sides divided by the perpendicular drawn to the third side from the vertex of the opposite angle.*

PROP. XXXI. THEOREM.

288. *In any triangle, the product of any two sides is equal to the product of the segments of the third side formed by the bisector of the opposite angle, plus the square of the bisector.*



Given, in $\triangle ABC$, line AD bisecting $\angle A$, meeting BC at D.

To Prove $AB \times AC = BD \times DC + \overline{AD}^2$.

Proof. Circumscribe a $\odot$ about $\triangle ABC$; produce AD to meet the circumference at E, and draw line CE.

Then in $\triangle$ ABD and ACE, by hyp.,

$$\angle BAD = \angle CAE.$$

Also, $\angle B = \angle E$,

since each is measured by $\frac{1}{2}$ arc AC. (?)

Then, $\triangle$ ABD and ACE are similar. (?)

In $\triangle$ ABD and ACE, sides AB and AE are homologous, as also are sides AD and AC. (§ 258)

$$\therefore AB : AD = AE : AC. \quad (?)$$

$$\therefore AB \times AC = AD \times AE \quad (?)$$

$$= AD \times (DE + AD)$$

$$= AD \times DE + \overline{AD}^2.$$

But $AD \times DE = BD \times DC$. (§ 280)

$$\therefore AB \times AC = BD \times DC + \overline{AD}^2.$$

EXERCISES.

21. The square of the altitude of an equilateral triangle is equal to three-fourths the square of the side.

22. If AD is the perpendicular from A to side BC of triangle ABC, prove

$$\overline{AB}^2 - \overline{AC}^2 = \overline{BD}^2 - \overline{CD}^2.$$

23. If one leg of a right triangle is double the other, the perpendicular from the vertex of the right angle to the hypotenuse divides it into segments which are to each other as 1 to 4. (§ 271.)

24. If two parallels to side BC of triangle ABC meet sides AB and AC at D and F, and E and G, respectively, prove

$$\frac{BD}{CE} = \frac{BF}{CG} = \frac{DF}{EG}. \quad (\S\ 247.)$$

25. C and D are respectively the middle points of a chord AB and its subtended arc. If $AD = 12$ and $CD = 8$, what is the diameter of the circle? (§ 271.)

26. If AD and BE are the perpendiculars from vertices A and B of triangle ABC to the opposite sides, prove

$$AC : DC = BC : EC.$$

(Prove $\triangle$ ACD and BCE similar.)

27. If D is the middle point of side BC of triangle ABC, right-angled at C, prove $\overline{AB}^2 - \overline{AD}^2 = 3\,\overline{CD}^2$.

28. The diameters of two concentric circles are 14 and 50 units, respectively. Find the length of a chord of the greater circle which is tangent to the smaller. (§ 273.)

29. The length of a tangent to a circle from a point 8 units distant from the nearest point of the circumference, is 12 units. What is the diameter of the circle?

(Let x represent the radius.)

30. The non-parallel sides AD and BC of trapezoid $ABCD$ intersect at O. If $AB = 15$, $CD = 24$, and the altitude of the trapezoid is 8, what is the altitude of triangle OAB? (§ 264.)

(Draw $CE \parallel AD$.)

31. If the equal sides of an isosceles right triangle are each 18 units in length, what is the length of the median drawn from the vertex of the right angle?

32. The non-parallel sides of a trapezoid are each 53 units in length, and one of the parallel sides is 56 units longer than the other. Find the altitude of the trapezoid.

33. AB is a chord of a circle, and CE is any chord drawn through the middle point C of arc AB, cutting chord AB at D. Prove AC a mean proportional between CD and CE.

(Prove ⨞ ACD and ACE similar.)

34. Two secants are drawn to a circle from an outside point. If their external segments are 12 and 9, respectively, while the internal segment of the former is 8, what is the internal segment of the latter? (§ 284.)

35. If, in triangle ABC, $\angle C = 120°$, prove

$$\overline{AB}^2 = \overline{BC}^2 + \overline{AC}^2 + AC \times BC.$$

(Fig. of Prop. XXVI. $\triangle ACD$ is one-half an equilateral $\triangle$.)

36. BC is the base of an isosceles triangle ABC inscribed in a circle. If a chord AD be drawn cutting BC at E, prove

$$AD : AB = AB : AE.$$

(Prove ⨞ ABD and ABE similar.)

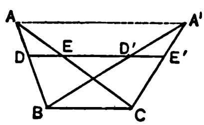
37. Two parallel chords on opposite sides of the centre of a circle are 48 units and 14 units long, respectively, and the distance between their middle points is 31 units. What is the diameter of the circle?

(Let x represent the distance from the centre to the middle point of one chord, and $31 - x$ the distance from the centre to the middle point of the other. Then the square of the radius may be expressed in two ways in terms of x.)

38. ABC is a triangle inscribed in a circle. Another circle is drawn tangent to the first externally at C, and AC and BC are produced to meet its circumference at D and E, respectively. Prove triangles ABC and CDE similar. (§ 197.)

(Draw a common tangent to the ⓢ at C. Then BC and CE are arcs of the same number of degrees.)

39. ABC and $A'BC$ are triangles whose vertices A and A' lie in a parallel to their common base BC. If a parallel to BC cuts AB and AC at D and E, and $A'B$ and $A'C$ at D' and E', respectively, prove $DE = D'E'$.

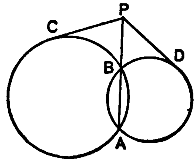


$\left(\text{Prove } \frac{DE}{BC} = \frac{D'E'}{BC}.\right)$

40. A line parallel to the bases of a trapezoid, passing through the intersection of the diagonals, and terminating in the non-parallel sides, is bisected by the diagonals. (Ex. 39.)

41. If the sides of triangle ABC are $AB = 10$, $BC = 14$, and $CA = 16$, find the lengths of the three medians. (§ 279, I.)

42. If the sides of a triangle are $AB = 4$, $AC = 5$, and $BC = 6$, find the length of the bisector of angle A. (§§ 249, 288.)



43. The tangents to two intersecting circles from any point in their common chord produced are equal. (§ 282.)

44. If two circles intersect, their common chord produced bisects their common tangents.

45. AB and AC are the tangents to a circle from point A. If CD be drawn perpendicular to radius OB at D, prove

$$AB : OB = BD : CD.$$

(Prove ▲ OAB and BCD similar by § 262.)

46. ABC is a triangle inscribed in a circle. A line AD is drawn from A to any point of BC, and a chord BE is drawn, making $\angle ABE = \angle ADC$. Prove

$$AB \times AC = AD \times AE.$$

(Prove $AB : AE = AD : AC$.)

47. The radius of a circle is $22\frac{1}{2}$ units. Find the length of a chord which joins the points of contact of two tangents, each 30 units in length, drawn to the circle from a point without the circumference.

(By § 271, 2, the radius is a mean proportional between the distances from the centre to the chord and to the point without the circumference; in this way the distance from the centre to the chord can be found.)

48. If, in right triangle ABC, acute angle B is double acute angle A, prove $\overline{AC}^2 = 3\,\overline{BC}^2$. (Ex. 104, p. 71.)

49. Find the product of the segments of any chord drawn through a point 9 units from the centre of a circle whose diameter is 24 units.

50. The hypotenuse of a right triangle is 5, and the perpendicular to it from the opposite vertex is $2\frac{2}{5}$. Find the legs, and the segments into which the perpendicular divides the hypotenuse. (§ 271.)

(Let x represent one of the segments of the hypotenuse.)

51. State and prove the converse of Prop. XIII.

(Fig. of Prop. XIII. To prove $\angle BAD = \angle CAD$. Produce CA to E, making $AE = AB$.)

52. State and prove the converse of Prop. XIV.

(Fig. of Prop. XIV. Lay off $AF = AB$.)

53. If D is the middle point of hypotenuse AB of right triangle ABC, prove

$$\overline{CD}^2 = \frac{1}{4}(\overline{AB}^2 + \overline{BC}^2 + \overline{CA}^2). \quad \text{(Ex. 83, p. 69.)}$$

54. If a line be drawn from vertex C of isosceles triangle ABC, meeting base AB produced at D, prove

$$\overline{CD}^2 - \overline{CB}^2 = AD \times BD. \quad (\S\,278.)$$

55. If AB is the base of isosceles triangle ABC, and AD be drawn perpendicular to BC, prove

$$3\,\overline{AD}^2 + \overline{BD}^2 + 2\,\overline{CD}^2 = \overline{AB}^2 + \overline{BC}^2 + \overline{CA}^2.$$

(We have $3\,\overline{AD}^2 = \overline{AD}^2 + 2\,\overline{AD}^2$.)

56. The middle points of two chords are distant 5 and 9 units, respectively, from the middle points of their subtended arcs. If the length of the first chord is 20 units, find the length of the second.

(Find the diameter by aid of § 270, 1.)

57. The sides AB and AC, of triangle ABC, are 16 and 9, respectively, and the length of the median drawn from C is 11. Find side BC. (§ 279, I.)

58. The diameter which bisects a chord whose length is $33\frac{3}{5}$ units, is 35 units in length. Find the distance from either extremity of the chord to the extremities of the diameter.

(Let x represent one segment of the diameter made by the chord.)

59. The equal angles of an isosceles triangle are each 30°, and the equal sides are each 8 units in length. What is the length of the base? (Ex. 104, p. 71.)

60. The diagonals of a trapezoid, whose bases are AD and BC, intersect at E. If $AE = 9$, $EC = 3$, and $BD = 16$, find BE and ED.

(△s AED and BEC are similar. Find BE by § 237.)

61. Prove the theorem of § 284 by drawing $A'B$ and AB'.

62. The parallel sides, AD and BC, of a circumscribed trapezoid are 18 and 6, respectively, and the other two sides are equal to each other. Find the diameter of the circle.

(Find AB by Ex. 31, p. 100. Draw through B a ∥ to CD.)

63. An angle of a triangle is acute, right, or obtuse according as the square of the opposite side is less than, equal to, or greater than, the sum of the squares of the other two sides.

(Prove by *Reductio ad Absurdum*.)

64. Is the greatest angle of a triangle whose sides are 3, 5, and 6, acute, right, or obtuse?

65. Is the greatest angle of a triangle whose sides are 8, 9, and 12, acute, right, or obtuse?

66. Is the greatest angle of a triangle whose sides are 12, 35, and 37, acute, right, or obtuse?

67. If two adjacent sides and one of the diagonals of a parallelogram are 7, 9, and 8, respectively, find the other diagonal.

(One-half of either diagonal is a median of the △ whose sides are, respectively, the given sides and the other diagonal of the ▱.)

68. If D is the intersection of the perpendiculars from the vertices of triangle ABC to the opposite sides, prove

$$\overline{AB}^2 - \overline{AC}^2 = \overline{BD}^2 - \overline{CD}^2. \quad (§ 272.)$$

69. If a parallel to hypotenuse AB of right triangle ABC meets AC and BC at D and E, respectively, prove

$$\overline{AE}^2 + \overline{BD}^2 = \overline{AB}^2 + \overline{DE}^2.$$

70. The diameters of two circles are 12 and 28, respectively, and the distance between their centres is 29. Find the length of the common tangent which cuts the straight line joining the centres.

(Find the ⊥ drawn from the centre of the smaller ⊙ to the radius of the greater ⊙ produced through the point of contact.)

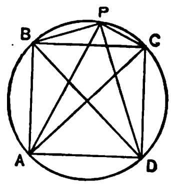
71. State and prove the converse of Prop. XXIII., III.

(Fig. of Prop. XXIII. ≜ ABC and ACD are similar.)

72. State and prove the converse of Prop. XXIII., II.

73. The sum of the squares of the distances of any point in the circumference of a circle from the vertices of an inscribed square, is equal to twice the square of the diameter of the circle. (§ 195.)

(To prove $\overline{PA}^2 + \overline{PB}^2 + \overline{PC}^2 + \overline{PD}^2 = 2\,\overline{AC}^2$.)



74. The sides AB, BC, and CA, of triangle ABC, are 13, 14, and 15, respectively. Find the segments into which AB and BC are divided by perpendiculars drawn from C and A, respectively.

(∠s BAC and ACB are acute by § 98. Find the segments by § 277.)

75. In right triangle ABC is inscribed a square $DEFG$, having its vertices D and G in hypotenuse BC, and its vertices E and F in sides AB and AC, respectively. Prove $BD : DE = DE : CG$.

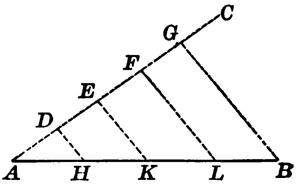
(Prove ≜ BDE and CFG similar.)

Note. For additional exercises on Book III., see p. 226.

CONSTRUCTIONS.

Prop. XXXII. Problem.

289. *To divide a given straight line into any number of equal parts.*



Given line AB.

Required to divide AB into four equal parts.

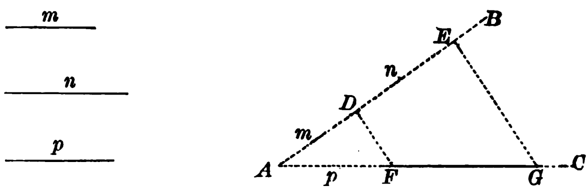
Construction. On the indefinite line AC, take any convenient length AD; on DC take $DE = AD$; on EC take $EF = AD$; on FC take $FG = AD$; and draw line BG.

Draw lines DH, EK, and $FL \parallel BG$, meeting AB at H, K, and L, respectively.

$\therefore AH = HK = KL = LB.$ (§ 242)

Prop. XXXIII. Problem.

290. *To construct a fourth proportional* (§ 231) *to three given straight lines.*



Given lines m, n, and p.

Required to construct a fourth proportional to m, n, and p.

Construction. Draw the indefinite lines AB and AC, making any convenient $\angle$ with each other.

On AB take $AD = m$; on DB take $DE = n$; on AC take $AF = p$.

Draw line DF; also, line $EG \parallel DF$, meeting AC at G.

Then, FG is a fourth proportional to m, n, and p.

Proof. Since DF is $\parallel$ to side EG of $\triangle AEG$,

$$AD : DE = AF : FG. \quad (?)$$

That is, $$m : n = p : FG.$$

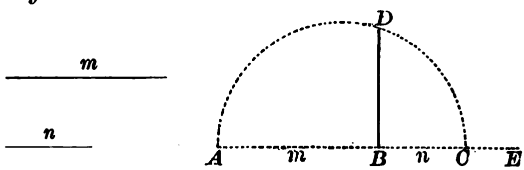
291. Cor. If we take $AF = n$, the proportion becomes

$$m : n = n : FG.$$

In this case, FG is a *third proportional* (§ 230) to m and n.

PROP. XXXIV. PROBLEM.

292. *To construct a mean proportional* (§ 230) *between two given straight lines.*



Given lines m and n.

Required to construct a mean proportional between m and n.

Construction. On the indefinite line AE, take $AB = m$; on BE take $BC = n$.

With AC as a diameter, describe the semi-circumference ADC.

Draw line $BD \perp AC$, meeting the arc at D.

Then, BD is a mean proportional between m and n.

(The proof is left to the pupil; see § 270.)

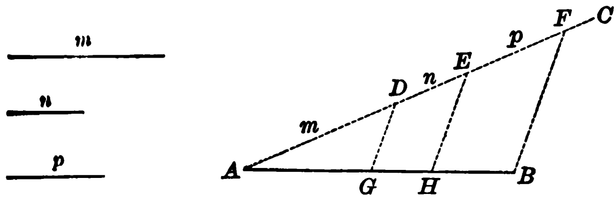
293. Sch. By aid of § 292, a line may be constructed equal to $\sqrt{a}$, where a is any number whatever.

Thus, to construct a line equal to $\sqrt{3}$, we take AB equal to 3 units, and BC equal to 1 unit.

Then, $BD = \sqrt{AB \times BC}$ (§ 232) $= \sqrt{3 \times 1} = \sqrt{3}$.

PROP. XXXV. PROBLEM.

294. *To divide a given straight line into parts proportional to any number of given lines.*



Given line AB, and lines m, n, and p.

Required to divide AB into parts proportional to m, n, and p.

Construction. On the indefinite line AC, take $AD = m$; on DC take $DE = n$; on EC take $EF = p$; and draw line BF.

Draw lines DG and EH ∥ to BF, meeting AB at G and H, respectively.

Then, AB is divided into parts AG, GH, and HB proportional to m, n, and p, respectively.

Proof. Since DG is ∥ to side EH of $\triangle AEH$,

$$\frac{AH}{AE} = \frac{AG}{AD} = \frac{GH}{DE}. \qquad (?)$$

That is,
$$\frac{AH}{AE} = \frac{AG}{m} = \frac{GH}{n}. \qquad (1)$$

And since EH is ∥ to side BF of $\triangle ABF$,

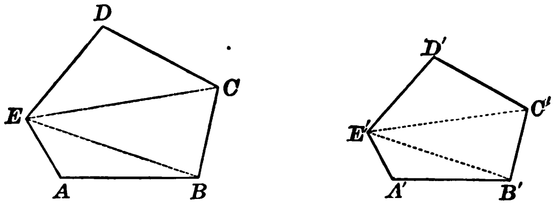
$$\frac{AH}{AE} = \frac{HB}{EF} = \frac{HB}{p}. \qquad (2)$$

From (1) and (2),
$$\frac{AG}{m} = \frac{GH}{n} = \frac{HB}{p}. \qquad (?)$$

Ex. 76. Construct a line equal to $\sqrt{2}$; to $\sqrt{5}$; to $\sqrt{6}$.

PROP. XXXVI. PROBLEM.

295. *Upon a given side, homologous to a given side of a given polygon, to construct a polygon similar to the given polygon.*



Given polygon $ABCDE$, and line $A'B'$.

Required to construct upon side $A'B'$, homologous to AB, a polygon similar to $ABCDE$.

Construction. Divide polygon $ABCDE$ into $\triangle$s by drawing diagonals EB and EC.

At A' construct $\angle B'A'E' = \angle A$; and draw line $B'E'$, making $\angle A'B'E' = \angle ABE$, meeting $A'E'$ at E'.

Then, $\triangle A'B'E'$ will be similar to $\triangle ABE$. (?)

In like manner, construct $\triangle B'C'E'$ similar to $\triangle BCE$, and $\triangle C'D'E'$ similar to $\triangle CDE$.

Then, polygon $A'B'C'D'E'$ will be similar to polygon $ABCDE$. (§ 266)

296. Def. A straight line is said to be divided by a given point in *extreme and mean ratio* when one of the segments (§ 250) is a mean proportional between the whole line and the other segment.

Thus, line AB is divided *internally* in extreme and mean ratio at C if

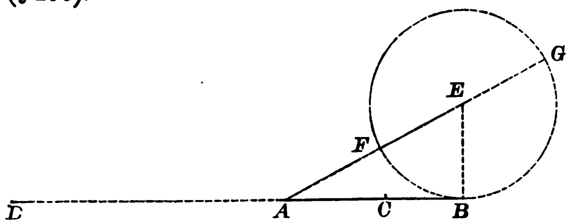
$$AB : AC = AC : BC;$$

and *externally* in extreme and mean ratio at D if

$$AB : AD = AD : BD.$$

PROP. XXXVII. PROBLEM.

297. *To divide a given straight line in extreme and mean ratio* (§ 296).



Given line AB.

Required to divide AB in extreme and mean ratio.

Construction. Draw line $BE \perp AB$, and equal to $\frac{1}{2} AB$.

With E as a centre and EB as a radius, describe $\odot BFG$.

Draw line AE cutting the circumference at F and G.

On AB take $AC = AF$; on BA produced, take $AD = AG$.

Then, AB is divided at C internally, and at D externally, in extreme and mean ratio.

Proof. Since AG is a secant, and AB a tangent,

$$AG : AB = AB : AF. \quad (\S\ 283)$$

$$\therefore\ AG : AB = AB : AC. \quad (1)$$

$$\therefore\ AG - AB : AB = AB - AC : AC. \quad (?)$$

$$\therefore\ AB : AG - AB = AC : BC. \quad (?)$$

But by cons., $AB = 2\,BE = FG$. (2)

$$\therefore\ AG - AB = AG - FG = AF = AC.$$

Substituting, $AB : AC = AC : BC$. (3)

Therefore, AB is divided at C internally in extreme and mean ratio.

Again, from (1),

$$AG + AB : AG = AB + AC : AB. \quad (?)$$

But, $AG + AB = AD + AB = BD$.

And by (2), $AB + AC = FG + AF = AG$.

$$\therefore BD : AG = AG : AB.$$

$$\therefore AB : AG = AG : BD. \qquad (?)$$

$$\therefore AB : AD = AD : BD.$$

Therefore, AB is divided at D externally in extreme and mean ratio.

298. Cor. If AB be denoted by m, and AC by x, proportion (3) of § 297 becomes

$$m : x = x : m - x.$$

$$\therefore x^2 = m(m - x) = m^2 - mx. \qquad (\text{§ } 232)$$

Or, $$x^2 + mx = m^2.$$

Multiplying by 4, and adding m^2 to both members,

$$4x^2 + 4mx + m^2 = 4m^2 + m^2 = 5m^2.$$

Extracting the square root of both members,

$$2x + m = \pm m\sqrt{5}.$$

Since x cannot be negative, we take the positive sign before the radical sign; then,

$$2x = m\sqrt{5} - m.$$

$$\therefore x (\text{or } AC) = \frac{m(\sqrt{5} - 1)}{2}.$$

EXERCISES.

77. To inscribe in a given circle a triangle similar to a given triangle. (§ 261.)

(Circumscribe a ⊙ about the given △, and draw radii to the vertices.)

78. To circumscribe about a given circle a triangle similar to a given triangle. (§ 262.)

Book IV.

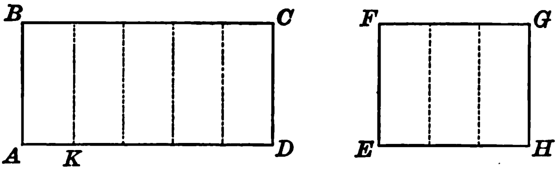
AREAS OF POLYGONS

Prop. I. Theorem.

299. *Two rectangles having equal altitudes are to each other as their bases.*

Note. The words "rectangle," "parallelogram," "triangle," etc., in the propositions of Book IV., mean the *amount of surface* in the rectangle, parallelogram, triangle, etc.

Case I. *When the bases are commensurable.*



Given rectangles $ABCD$ and $EFGH$, with equal altitudes AB and EF, and commensurable bases AD and EH.

To Prove $\frac{ABCD}{EFGH} = \frac{AD}{EH}$.

Proof. Let AK be a common measure of AD and EH, and let it be contained 5 times in AD, and 3 times in EH.

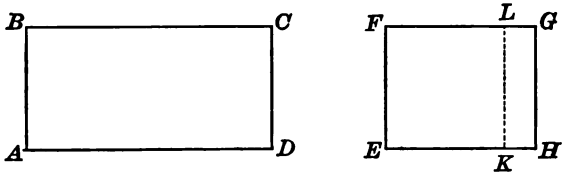
$$\therefore \frac{AD}{EH} = \frac{5}{3}. \qquad (1)$$

Drawing ⊥s to AD and EH through the several points of division, rect. $ABCD$ will be divided into 5 parts, and rect. $EFGH$ into 3 parts, all of which parts are equal. (§ 114)

$$\therefore \frac{ABCD}{EFGH} = \frac{5}{3}. \qquad (2)$$

From (1) and (2), $\frac{ABCD}{EFGH} = \frac{AD}{EH}$. (?)

Case II. *When the bases are incommensurable.*



Given rectangles $ABCD$ and $EFGH$, with equal altitudes AB and EF, and incommensurable bases AD and EH.

To Prove $$\frac{ABCD}{EFGH}=\frac{AD}{EH}.$$

Proof. Divide AD into any number of equal parts, and apply one of these parts to EH as a unit of measure.

Since AD and EH are incommensurable, a certain number of the parts will extend from E to K, leaving a remainder $KH <$ one of the equal parts.

Draw line $KL \perp EH$, meeting FG at L.

Then, since AD and EK are commensurable,

$$\frac{ABCD}{EFLK}=\frac{AD}{EK}. \qquad \text{(§ 299, Case I.)}$$

Now let the number of subdivisions of AD be indefinitely increased.

Then the unit of measure will be indefinitely diminished, and the remainder KH will approach the limit 0.

Then, $\frac{ABCD}{EFLK}$ will approach the limit $\frac{ABCD}{EFGH}$,

and $\frac{AD}{EK}$ will approach the limit $\frac{AD}{EH}$.

By the Theorem of Limits, these limits are equal. (?)

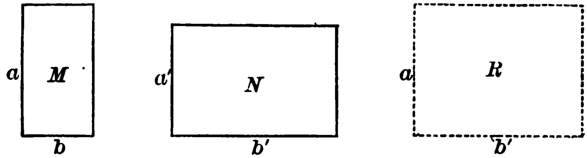
$$\therefore \frac{ABCD}{EFGH}=\frac{AD}{EH}.$$

300. Cor. Since either side of a rectangle may be taken as the base, it follows that

Two rectangles having equal bases are to each other as their altitudes.

PROP. II. THEOREM.

301. *Any two rectangles are to each other as the products of their bases by their altitudes.*



Given M and N rectangles, with altitudes a and a', and bases b and b', respectively.

To Prove $$\frac{M}{N}=\frac{a \times b}{a' \times b'}.$$

Proof. Let R be a rect. with altitude a and base b'.

Then, since rectangles M and R have equal altitudes, they are to each other as their bases. (§ 299)

$$\therefore \frac{M}{R}=\frac{b}{b'}. \quad (1)$$

And since rectangles R and N have equal bases, they are to each other as their altitudes. (?)

$$\therefore \frac{R}{N}=\frac{a}{a'}. \quad (2)$$

Multiplying (1) and (2), we have

$$\frac{M}{R}\times\frac{R}{N}, \text{ or } \frac{M}{N}=\frac{a \times b}{a' \times b'}.$$

DEFINITIONS.

302. The *area* of a surface is its ratio to another surface, called the *unit of surface*, adopted arbitrarily as the unit of measure (§ 180).

The usual unit of surface is a square whose side is some *linear unit;* for example, a *square inch* or a *square foot.*

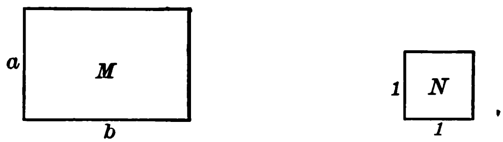
303. Two surfaces are said to be *equivalent* (≎), when their areas are equal.

304. The *dimensions* of a rectangle are its base and altitude.

PROP. III. THEOREM.

305. *The area of a rectangle is equal to the product of its base and altitude.*

Note. In all propositions relating to areas, the unit of surface (§ 302) is understood to be *a square whose side is the linear unit.*



Given a and b, the altitude and base, respectively, of rect. M; and N the unit of surface, *i.e.*, a square whose side is the linear unit.

To Prove that, if N is the unit of surface,

$$\text{area } M = a \times b.$$

Proof. Since any two rectangles are to each other as the products of their bases by their altitudes (§ 301),

$$\frac{M}{N} = \frac{a \times b}{1 \times 1} = a \times b.$$

But since N is the unit of surface, the ratio of M to N is the *area* of M. (§ 302)

$$\therefore \text{ area } M = a \times b.$$

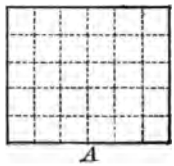
306. Sch. I. The statement of Prop. III. is an abbreviation of the following:

If the unit of surface is a square whose side is the linear unit, the *number* which expresses the area of a rectangle is equal to the product of the *numbers* which express the lengths of its sides.

An interpretation of this form is always understood in every proposition relating to areas.

307. Cor. *The area of a square is equal to the square of its side.*

308. Sch. II. If the sides of a rectangle are *multiples* of the linear unit, the truth of Prop. III. may be seen by dividing the figure into squares, each equal to the unit of surface.

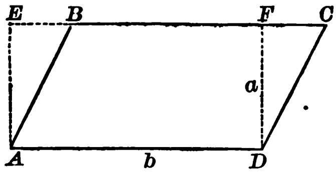


Thus, if the altitude of rectangle A is 5 units, and its base 6 units, the figure can be divided into 30 squares.

In this case, 30, the number which expresses the area of the rectangle, is the product of 6 and 5, the numbers which express the lengths of the sides.

Prop. IV. Theorem.

309. *The area of a parallelogram is equal to the product of its base and altitude.*



Given ▱ $ABCD$, with its altitude $DF = a$, and its base $AD = b$.

To Prove area $ABCD = a \times b$.

Proof. Draw line $AE \parallel DF$, meeting CB produced at E.

Then, $AEFD$ is a rectangle. (?)

In rt. ⊿s ABE and DCF,

$AB = DC$, and $AE = DF$. (?)

$\therefore \triangle ABE = \triangle DCF$. (?)

Now if from the entire figure $ADCE$ we take $\triangle ABE$, there remains ▱ $ABCD$; and if we take $\triangle DCF$, there remains rect. $AEFD$.

$\therefore$ area $ABCD =$ area $AEFD = a \times b$. (§ 305)

310. Cor. I. *Two parallelograms having equal bases and equal altitudes are equivalent* (§ 303).

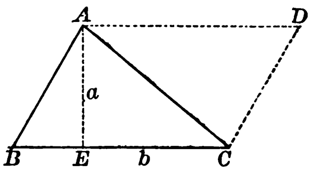
311. Cor. II. 1. *Two parallelograms having equal altitudes are to each other as their bases.*

2. *Two parallelograms having equal bases are to each other as their altitudes.*

3. *Any two parallelograms are to each other as the products of their bases by their altitudes.*

PROP. V. THEOREM.

312. *The area of a triangle is equal to one-half the product of its base and altitude.*



Given $\triangle ABC$, with its altitude $AE = a$, and its base $BC = b$.

To Prove area $ABC = \frac{1}{2} a \times b$.

(By § 108, AC divides $\square ABCD$ into two equal $\triangle$s.)

313. Cor. I. *Two triangles having equal bases and equal altitudes are equivalent.*

314. Cor. II. 1. *Two triangles having equal altitudes are to each other as their bases.*

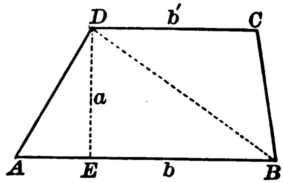
2. *Two triangles having equal bases are to each other as their altitudes.*

3. *Any two triangles are to each other as the products of their bases by their altitudes.*

315. Cor. III. *A triangle is equivalent to one-half of a parallelogram having the same base and altitude.*

PROP. VI. THEOREM.

316. *The area of a trapezoid is equal to one-half the sum of its bases multiplied by its altitude.*



Given trapezoid $ABCD$, with its altitude DE equal to a, and its bases AB and DC equal to b and b', respectively.

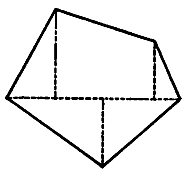
To Prove area $ABCD = a \times \frac{1}{2}(b + b')$.

(The trapezoid is composed of two $\triangle$s whose altitude is a, and bases b and b', respectively.)

317. Cor. Since the line joining the middle points of the non-parallel sides of a trapezoid is equal to one-half the sum of the bases (§ 132), it follows that

The area of a trapezoid is equal to the product of its altitude by the line joining the middle points of its non-parallel sides.

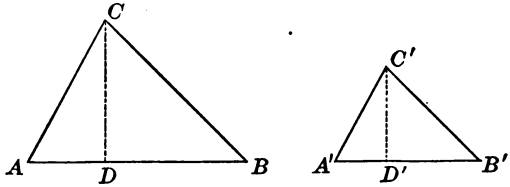
318. Sch. The area of any polygon may be obtained by finding the sum of the areas of the triangles into which the polygon may be divided by drawing diagonals from any one of its vertices.



But in practice it is better to draw the longest diagonal, and draw perpendiculars to it from the remaining vertices of the polygon. The polygon will then be divided into right triangles and trapezoids; and by measuring the lengths of the perpendiculars, and of the portions of the diagonal which they intercept, the areas of the figures may be found by §§ 312 and 316.

PROP. VII. THEOREM.

319. *Two similar triangles are to each other as the squares of their homologous sides.*



Given AB and $A'B'$ homologous sides of similar △ ABC and $A'B'C'$, respectively.

To Prove $$\frac{ABC}{A'B'C'} = \frac{\overline{AB}^2}{\overline{A'B'}^2}.$$

Proof. Draw altitudes CD and $C'D'$.

$$\therefore \frac{ABC}{A'B'C'} = \frac{AB \times CD}{A'B' \times C'D'} \quad (\S\ 314, 3)$$

$$= \frac{AB}{A'B'} \times \frac{CD}{C'D'}. \quad (1)$$

But, $$\frac{CD}{C'D'} = \frac{AB}{A'B'}. \quad (\S\ 264)$$

Substituting this value in (1),

$$\frac{ABC}{A'B'C'} = \frac{AB}{A'B'} \times \frac{AB}{A'B'} = \frac{\overline{AB}^2}{\overline{A'B'}^2}.$$

320. Sch. Two similar triangles are to each other as the squares of any two homologous lines.

EXERCISES.

1. If the area of a rectangle is 7956 sq. in., and its base $3\frac{1}{4}$ yd., find its perimeter in feet.

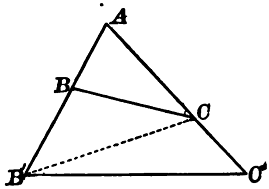
2. If the base and altitude of a rectangle are 14 ft. 7 in., and 5 ft. 3 in., respectively, what is the side of an equivalent square?

3. Find the dimensions of a rectangle whose area is 168, and perimeter 52.

(Let x represent the base.)

PROP. VIII. THEOREM.

321. *Two triangles having an angle of one equal to an angle of the other, are to each other as the products of the sides including the equal angles.*



Given $\angle A$ common to $\triangle ABC$ and $AB'C'$.

To Prove $$\frac{ABC}{AB'C'} = \frac{AB \times AC}{AB' \times AC'}.$$

Proof. Draw line $B'C$.

Then $\triangle ABC$ and $AB'C$, having the common vertex C, and their bases AB and AB' in the same str. line, have the same altitude.

$$\therefore \frac{ABC}{AB'C} = \frac{AB}{AB'}. \qquad (\S\ 314, 1)$$

And $\triangle AB'C$ and $AB'C'$, having the common vertex B, and their bases AC and AC' in the same str. line, have the same altitude.

$$\therefore \frac{AB'C}{AB'C'} = \frac{AC}{AC'}.$$

Multiplying these equations, we have

$$\frac{ABC}{AB'C} \times \frac{AB'C}{AB'C'}, \text{ or } \frac{ABC}{AB'C'} = \frac{AB \times AC}{AB' \times AC'}.$$

EXERCISES.

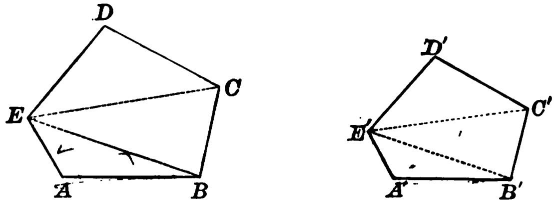
4. The area of a rectangle is 143 sq. ft. 75 sq. in., and its base is 3 times its altitude. Find each of its dimensions.

(Let x represent the altitude.)

5. The hypotenuse of a right triangle is 5 ft. 5 in., and one of its legs is 2 ft. 9 in. Find its area.

PROP. IX. THEOREM.

322. *Two similar polygons are to each other as the squares of their homologous sides.*



Given AB and $A'B'$ homologous sides of similar polygons AC and $A'C'$, whose areas are K and K', respectively.

To Prove $$\frac{K}{K'} = \frac{\overline{AB}^2}{\overline{A'B'}^2}.$$

Proof. Draw diagonals EB, EC, $E'B'$, and $E'C'$.

Then, $\triangle ABE$ is similar to $\triangle A'B'E'$. (§ 267)

$$\therefore \frac{ABE}{A'B'E'} = \frac{\overline{AB}^2}{\overline{A'B'}^2}. \qquad (§ 319)$$

In like manner,

$$\frac{BCE}{B'C'E'} = \frac{\overline{BC}^2}{\overline{B'C'}^2} = \frac{\overline{AB}^2}{\overline{A'B'}^2},$$

and $$\frac{CDE}{C'D'E'} = \frac{\overline{CD}^2}{\overline{C'D'}^2} = \frac{\overline{AB}^2}{\overline{A'B'}^2}. \qquad (§ 253, 2)$$

$$\therefore \frac{ABE}{A'B'E'} = \frac{BCE}{B'C'E'} = \frac{CDE}{C'D'E'}. \qquad (?)$$

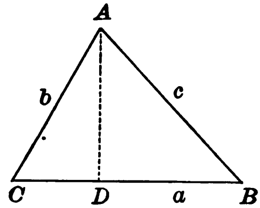
$$\therefore \frac{ABE + BCE + CDE}{A'B'E' + B'C'E' + C'D'E'} = \frac{ABE}{A'B'E'}. \qquad (§ 240)$$

$$\therefore \frac{K}{K'} = \frac{ABE}{A'B'E'} = \frac{\overline{AB}^2}{\overline{A'B'}^2}.$$

323. Cor. *Two similar polygons are to each other as the squares of their perimeters.* (§ 268)

PROP. X. PROBLEM.

324. *To express the area of a triangle in terms of its three sides.*



Given sides BC, CA, and AB, of $\triangle ABC$, equal to a, b, and c, respectively.

Required to express area ABC in terms of a, b, and c.

Solution. Let C be an acute $\angle$, and draw altitude AD.

$$\therefore c^2 = a^2 + b^2 - 2a \times CD. \quad (\S\ 277)$$

Transposing, $2a \times CD = a^2 + b^2 - c^2$.

$$\therefore CD = \frac{a^2 + b^2 - c^2}{2a}.$$

$$\therefore \overline{AD}^2 = \overline{AC}^2 - \overline{CD}^2 \quad (\S\ 273)$$

$$= (AC + CD)(AC - CD)$$

$$= \left(b + \frac{a^2 + b^2 - c^2}{2a}\right)\left(b - \frac{a^2 + b^2 - c^2}{2a}\right)$$

$$= \frac{(2ab + a^2 + b^2 - c^2)(2ab - a^2 - b^2 + c^2)}{4a^2}$$

$$= \frac{[(a + b)^2 - c^2][c^2 - (a - b)^2]}{4a^2}$$

$$= \frac{(a + b + c)(a + b - c)(c + a - b)(c - a + b)}{4a^2}. \quad (1)$$

Now let $a + b + c = 2s$.

$$\therefore \overline{AD}^2 = \frac{2s(2s - 2c)(2s - 2b)(2s - 2a)}{4a^2}$$

$$= \frac{16s(s - a)(s - b)(s - c)}{4a^2}.$$

$$\therefore\ AD = \frac{2\sqrt{s(s-a)(s-b)(s-c)}}{a}.$$

$$\therefore\ \text{area } ABC = \tfrac{1}{2}a \times AD \qquad (?)$$

$$= \sqrt{s(s-a)(s-b)(s-c)}.$$

325. Sch. Let it be required to find the area of a triangle whose sides are 13, 14, and 15.

Let $a = 13$, $b = 14$, and $c = 15$; then

$$s = \tfrac{1}{2}(13 + 14 + 15) = 21.$$

Whence, $s - a = 8$, $s - b = 7$, and $s - c = 6$.

Then, the area of the triangle is

$$\sqrt{21 \times 8 \times 7 \times 6} = \sqrt{3 \times 7 \times 2^3 \times 7 \times 2 \times 3}$$
$$= \sqrt{2^4 \times 3^2 \times 7^2} = 2^2 \times 3 \times 7 = 84.$$

EXERCISES.

6. Find the area of a triangle whose sides are 8, 13, and 15.

7. The area of a square is 693 sq. yd. 4 sq. ft.; find its side.

8. If the altitude of a trapezoid is 1 ft. 4 in., and its bases 1 ft. 1 in. and 2 ft. 5 in., respectively, what is its area?

9. If, in figure of Prop. VII., $AB = 9$, $A'B' = 7$, and the area of $A'B'C'$ is 147, find area ABC.

10. If the sides of triangle ABC are $AB = 25$, $BC = 17$, and $CA = 28$, find its area, and the length of the perpendicular from each vertex to the opposite side.

11. Find the length of the diagonal of a rectangle whose area is 2640, and altitude 48.

12. Find the lower base of a trapezoid whose area is 9408, upper base 79, and altitude 96.

13. The area of a rhombus is equal to one-half the product of its diagonals. (§ 117.)

14. The diagonals of a parallelogram divide it into four equivalent triangles.

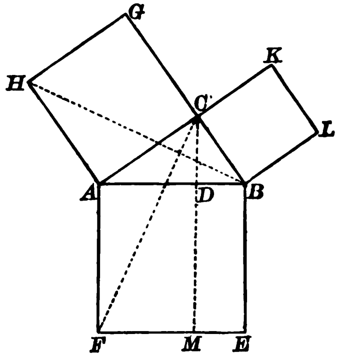
15. Lines drawn to the vertices of a parallelogram from any point in one of its diagonals divide the figure into two pairs of equivalent triangles. (Ex. 63, p. 67.)

16. The area of a certain triangle is $2\frac{1}{4}$ times the area of a similar triangle. If the altitude of the first triangle is 4 ft. 3 in., what is the homologous altitude of the second? (§ 320.)

326. Sch. Since the area of a square is equal to the square of its side (§ 307), we may state Prop. XXIV., Book III., as follows:

In any right triangle, the square described upon the hypotenuse is equivalent to the sum of the squares described upon the legs.

The theorem in the above form may be proved as follows:



Given $ABEF$, $ACGH$, and $BCKL$ squares described upon hypotenuse AB, and legs AC and BC, respectively, of rt. $\triangle ABC$.

To Prove area $ABEF =$ area $ACGH +$ area $BCKL$.

Proof. Draw line $CD \perp AB$, and produce it to meet EF at M; also, draw lines BH and CF.

Then in $\triangle$s ABH and ACF, by hyp.,

$$AB = AF \text{ and } AH = AC.$$

Also, $\angle BAH = \angle CAF$,

for each is equal to a rt. $\angle + \angle BAC$.

$$\therefore \triangle ABH = \triangle ACF. \qquad (?)$$

Now $\triangle ABH$ has the same base and altitude as square $ACGH$.

$$\therefore \text{area } ABH = \tfrac{1}{2} \text{area } ACGH. \qquad (\S\ 315)$$

And $\triangle ACF$ has the same base and altitude as rect. $ADMF$.

$\therefore$ area $ACF = \frac{1}{2}$ area $ADMF$.

But, area $ABH =$ area ACF.

$\therefore \frac{1}{2}$ area $ACGH = \frac{1}{2}$ area $ADMF$, (?)

or area $ACGH =$ area $ADMF$. (1)

Similarly, by drawing lines AL and CE, we may prove

area $BCKL =$ area $BDME$. (2)

Adding (1) and (2), we have

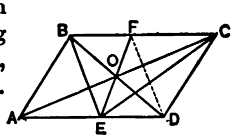
area $ACGH +$ area $BCKL =$ area $ABEF$.

327. Sch. The theorem of § 326 is supposed to have been first given by Pythagoras, and is called after him the *Pythagorean Theorem*.

Several other propositions of Book III. may be put in the form of statements in regard to areas; as, for example, Props. XXV. and XXVI.

EXERCISES.

17. If EF is any straight line drawn through the centre of parallelogram $ABCD$, meeting sides AD and BC at E and F, respectively, prove triangles BEF and CED equivalent. (Ex. 61, p. 66.)

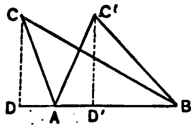


(Prove $BEDF$ a ▱ by § 112.)

18. The side of an equilateral triangle is 5; find its area. (Ex. 21, p. 151.)

19. The altitude of an equilateral triangle is 3; find its area.

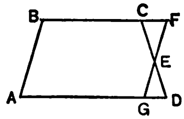
20. Two triangles are equivalent if they have two sides of one equal respectively to two sides of the other, and the included angles supplementary.



21. One diagonal of a rhombus is five-thirds of the other, and the difference of the diagonals is 8; find its area. (Ex. 13, p. 173.)

22. If D and E are the middle points of sides BC and AC, respectively, of triangle ABC, prove triangles ABD and ABE equivalent. (§ 80.)

23. If E is the middle point of CD, one of the non-parallel sides of trapezoid $ABCD$, and a parallel to AB drawn through E meets BC at F and AD at G, prove parallelogram $ABFG$ equivalent to the trapezoid.



24. The sides AB, BC, CD, and DA of quadrilateral $ABCD$ are 10, 17, 13, and 20, respectively, and the diagonal AC is 21. Find the area of the quadrilateral.

25. Find the area of the square inscribed in a circle whose radius is 3.

(The diagonal is a diameter, by § 157.)

26. The area of an isosceles right triangle is 81 sq. in.; find its hypotenuse in feet.

(Represent one of the equal sides by x.)

27. The area of an equilateral triangle is $9\sqrt{3}$; find its side.

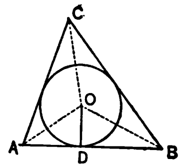
(Represent the side by x.)

28. The area of an equilateral triangle is $16\sqrt{3}$; find its altitude.

(Represent the altitude by x.)

29. The base of an isosceles triangle is 56, and each of the equal sides is 53; find its area.

30. The area of a triangle is equal to one-half the product of its perimeter by the radius of the inscribed circle.



31. The area of an isosceles right triangle is equal to one-fourth the area of the square described upon the base. (§ 307.)

32. If angle A of triangle ABC is 30°, prove

$$\text{area } ABC = \tfrac{1}{4} AB \times AC.$$

(Draw $CD \perp AB$; then CD may be found by Ex. 104, p. 71.)

33. A circle whose diameter is 12 is inscribed in a quadrilateral whose perimeter is 50. Find the area of the quadrilateral.

(Compare Ex. 30, p. 176.)

34. Two similar triangles have homologous sides equal to 8 and 15, respectively. Find the homologous side of a similar triangle equivalent to their sum. (§ 319.)

35. If E is any point within parallelogram $ABCD$, triangles ABE and CDE are together equivalent to one-half the parallelogram.

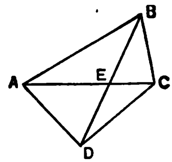
(Draw through E a $\parallel$ to AB.)

36. The non-parallel sides, AB and CD, of a trapezoid are each 25 units in length, and the sides AD and BC are 33 and 19 units, respectively. Find the area of the trapezoid.

(Draw through B a $\parallel$ to CD, and a $\perp$ to AD.)

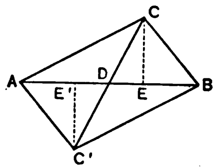
37. If the area of a polygon, one of whose sides is 15 in., is 375 sq. in., what is the area of a similar polygon whose homologous side is 18 in.?

38. If the area of a polygon, one of whose sides is 36 ft., is 648 sq. ft., what is the homologous side of a similar polygon whose area is 392 sq. ft.?



39. If one diagonal of a quadrilateral bisects the other, it divides the quadrilateral into two equivalent triangles.

(To prove $\triangle ABC \approx \triangle ACD$.)



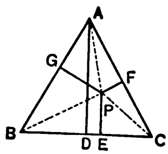
40. Two equivalent triangles have a common base, and lie on opposite sides of it. Prove that the base, produced if necessary, bisects the line joining their vertices.

(To prove $CD = C'D$.)

41. If the sides of a triangle are 15, 41, and 52, find the radius of the inscribed circle. (Ex. 30, p. 176.)

42. The area of a rhombus is 240, and its side is 17; find its diagonals. (Ex. 13, p. 173.)

(Represent the diagonals by $2x$ and $2y$.)



43. The sum of the perpendiculars from any point within an equilateral triangle to the three sides is equal to the altitude of the triangle.

44. The longest sides of two similar polygons are 18 and 3, respectively. How many polygons, each equal to the second, will form a polygon equivalent to the first? (§ 322.)

45. If the sides of a triangle are 25, 29, and 36, find the diameter of the circumscribed circle. (§ 287.)

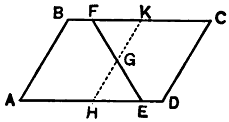
(The altitude of a $\triangle$ equals its area divided by one-half its base.)

46. If a is the base, and b one of the equal sides of an isosceles triangle, prove its area equal to $\frac{1}{4}a\sqrt{4b^2 - a^2}$.

47. The sides AB and AC of triangle ABC are 15 and 22, respectively. From a point D in AB, a parallel to BC is drawn meeting AC at E, and dividing the triangle into two equivalent parts. Find AD and AE. (§ 319.)

48. The segments of the hypotenuse of a right triangle made by a perpendicular drawn from the vertex of the right angle, are $5\frac{2}{5}$ and $9\frac{3}{5}$, respectively; find the area of the triangle.

49. Any straight line drawn through the centre of a parallelogram, terminating in a pair of opposite sides, divides the parallelogram into two equivalent quadrilaterals. (Ex. 61, p. 66.)



50. If E is the middle point of CD, one of the non-parallel sides of trapezoid $ABCD$, prove triangle ABE equivalent to $\frac{1}{2}ABCD$.

(Draw through E a $\parallel$ to AB.)

51. The sides of triangle ABC are $AB = 13$, $BC = 14$, and $CA = 15$. If AD is the bisector of angle A, meeting BC at D, find the areas of triangles ABD and ACD. (§§ 249, 325.)

52. The longest diagonal AD of pentagon $ABCDE$ is 44, and the perpendiculars to it from B, C, and E are 24, 16, and 15, respectively. If $AB = 25$, $CD = 20$, and $AE = 17$, what is the area of the pentagon? (§ 318.)

53. The sides of a triangle are proportional to the numbers 7, 24, and 25, respectively. The perpendicular to the third side from the vertex of the opposite angle is $13\frac{11}{25}$. Find the area of the triangle.

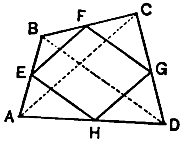
(Represent the sides by $7x$, $24x$, and $25x$, respectively; the $\triangle$ is a rt. $\triangle$ by Ex. 63, p. 154.)

54. If E and F are the middle points of sides AB and AC, respectively, of a triangle, and D is any point in BC, prove quadrilateral $AEDF$ equivalent to one-half triangle ABC.

(Prove $\triangle DEF \approx \frac{1}{2} \triangle ABC$, by aid of Ex. 64, p. 67.)

55. If E, F, G, and H are the middle points of sides AB, BC, CD, and DA, respectively, of quadrilateral $ABCD$, prove $EFGH$ a parallelogram equivalent to one-half $ABCD$.

(By Ex. 64, p. 67, area $EBF = \frac{1}{4}$ area ABC.)

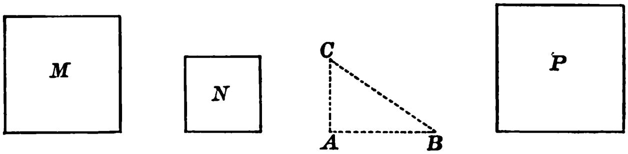


Note. For additional exercises on Book IV., see p. 229.

CONSTRUCTIONS.

PROP. XI. PROBLEM.

328. *To construct a square equivalent to the sum of two given squares.*



Given squares M and N.

Required to construct a square ≎ $M + N$.

Construction. Draw line AB equal to a side of M.

At A draw line $AC \perp AB$, and equal to a side of N; and draw line BC.

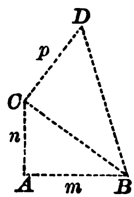
Then, square P, described with its side equal to BC, will be ≎ $M + N$.

Proof. In rt. $\triangle ABC$, $\overline{BC}^2 = \overline{AB}^2 + \overline{AC}^2$. (?)

$\therefore$ area $P =$ area $M +$ area N. (§ 307)

329. Cor. By an extension of the above method, a square may be constructed equivalent to the sum of any number of given squares.

Given three squares whose sides are equal to m, n, and p, respectively.



Required to construct a square ≎ the sum of the given squares.

Construction. Draw line $AB = m$.

Draw line $AC \perp AB$, and equal to n, and line BC.

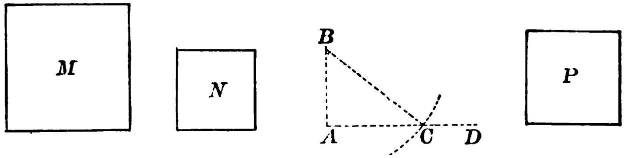
Draw line $CD \perp BC$, and equal to p, and line BD.

Then, the square described with its side equal to BD will be ≎ the sum of the given squares.

(The proof is left to the pupil.)

PROP. XII. PROBLEM.

330. *To construct a square equivalent to the difference of two given squares.*



Given squares M and N, M being $> N$.

Required to construct a square ≎ $M - N$.

Proof. Draw the indefinite line AD.

At A draw line $AB \perp AD$, and equal to a side of N.

With B as a centre, and with a radius equal to a side of M, describe an arc cutting AD at C, and draw line BC.

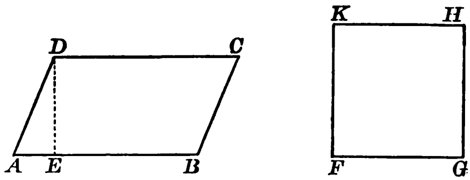
Then, square P, described with its side equal to AC, will be ≎ $M - N$.

Proof. In rt. $\triangle ABC$, $\overline{AC}^2 = \overline{BC}^2 - \overline{AB}^2$. (?)

$\therefore$ area $P =$ area $M -$ area N. (?)

PROP. XIII. PROBLEM.

331. *To construct a square equivalent to a given parallelogram.*



Given ▱ $ABCD$.

Required to construct a square ≎ $ABCD$.

Construction. Draw line $DE \perp AB$, and construct line FG a mean proportional between lines AB and DE (§ 292).

Then, square $FGHK$, described with its side equal to FG, will be ≎ $ABCD$.

Proof. By cons., $AB : FG = FG : DE.$

$$\therefore \overline{FG}^2 = AB \times DE. \qquad (?)$$

$$\therefore \text{area } FGHK = \text{area } ABCD. \qquad (?)$$

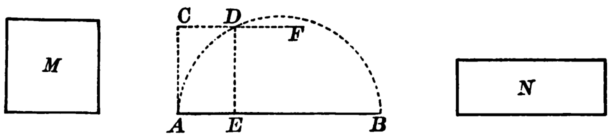
332. Cor. *A square may be constructed equivalent to a given triangle by taking for its side a mean proportional between the base and one-half the altitude of the triangle.*

Ex. 56. To construct a triangle equivalent to a given square, having given its base and an angle adjacent to the base.

(Take for the required altitude a third proportional to one-half the given base and the side of the given square.)

Prop. XIV. Problem.

333. *To construct a rectangle equivalent to a given square, having the sum of its base and altitude equal to a given line.*



Given square M, and line AB.

Required to construct a rectangle ⇔ M, having the sum of its base and altitude equal to AB.

Construction. With AB as a diameter, describe semicircumference ADB.

Draw line $AC \perp AB$, and equal to a side of M.

Draw line $CF \parallel AB$, intersecting arc ADB at D, and line $DE \perp AB$.

Then, rectangle N, constructed with its base and altitude equal to BE and AE, respectively, will be ⇔ M.

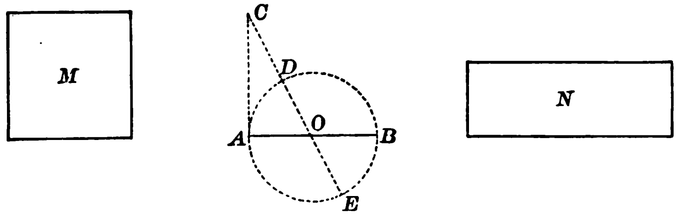
Proof. $AE : DE = DE : BE.$ (§ 270, 1)

$$\therefore AE \times BE = \overline{DE}^2 = \overline{AC}^2. \qquad (?)$$

$$\therefore \text{area } N = \text{area } M. \qquad (?)$$

Prop. XV. Problem.

334. *To construct a rectangle equivalent to a given square, having the difference of its base and altitude equal to a given line.*



Given square M, and line AB.

Required to construct a rectangle ≎ M, having the difference of its base and altitude equal to AB.

Construction. With AB as a diameter, describe ⊙ ADB.

Draw line $AC \perp AB$, and equal to a side of M.

Through centre O draw line CO, intersecting the circumference at D and E.

Then, rectangle N, constructed with its base and altitude equal to CE and CD, respectively, will be ≎ M.

Proof. $CE - CD = DE = AB$. (?)

That is, the difference of the base and altitude of N is equal to AB.

Again, AC is tangent to ⊙ ADB at A. (?)

$\therefore CD \times CE = \overline{CA}^2$. (§ 282)

$\therefore$ area $N =$ area M. (?)

EXERCISES.

57. To construct a triangle equivalent to a given triangle, having given its base.

(Take for the required altitude a fourth proportional to the given base, and the base and altitude of the given $\triangle$.)

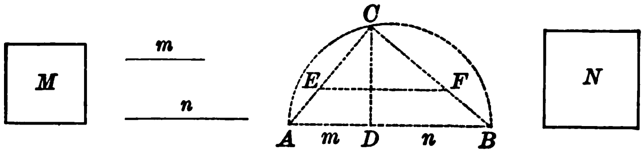
How many different $\triangle$s can be constructed?

58. To construct a rectangle equivalent to a given rectangle, having given its base.

59. To construct a square equivalent to twice a given square. (§ 307.)

PROP. XVI. PROBLEM.

335. *To construct a square having a given ratio to a given square.*



Given square M, and lines m and n.

Required to construct a square having to M the ratio $n : m$.

Construction. On line AB, take $AD = m$ and $DB = n$.

With AB as a diameter, describe semi-circumference ACB.

Draw line $DC \perp AB$, meeting arc ACB at C, and lines AC and BC.

On AC take CE equal to a side of M; and draw line $EF \parallel AB$, meeting BC at F.

Then, square N, constructed with its side equal to CF, will have to M the ratio $n : m$.

Proof. $\angle ACB$ is a rt. $\angle$. (?)

Then since CD is $\perp AB$,

$$\frac{\overline{AC}^2}{\overline{BC}^2} = \frac{AB \times AD}{AB \times BD} = \frac{AD}{BD} = \frac{m}{n}. \qquad (§\ 271, 2)$$

But since EF is $\parallel AB$,

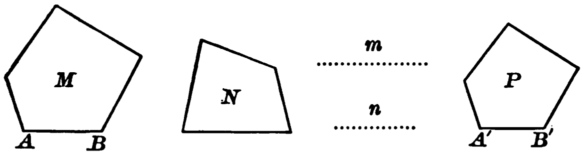
$$\frac{CE}{CF} = \frac{AC}{BC}. \qquad (?)$$

$$\therefore \frac{\overline{CE}^2}{\overline{CF}^2} = \frac{\overline{AC}^2}{\overline{BC}^2} = \frac{m}{n}.$$

$$\therefore \frac{\text{area } M}{\text{area } N} = \frac{m}{n}. \qquad (?)$$

PROP. XIX. PROBLEM.

338. *To construct a polygon similar to one of two given polygons, and equivalent to the other.*



Given polygons M and N.

Required to construct a polygon similar to M, and $\approx N$.

Construction. Let AB be any side of M.

Construct m, the side of a square $\approx M$, and n, the side of a square $\approx N$. (Note, p. 185)

Construct $A'B'$, a fourth proportional to m, n, and AB.

Upon side $A'B'$, homologous to AB, construct polygon P similar to M. (§ 295)

Then, $P \approx N$.

Proof. Since M is similar to P,

$$\frac{\text{area } M}{\text{area } P} = \frac{\overline{AB}^2}{\overline{A'B'}^2}. \quad (?)$$

But by cons., $m : n = AB : A'B'$, or $\dfrac{AB}{A'B'} = \dfrac{m}{n}$.

$$\therefore \frac{\text{area } M}{\text{area } P} = \frac{m^2}{n^2} = \frac{\text{area } M}{\text{area } N}. \quad (?)$$

$$\therefore \text{ area } P = \text{area } N.$$

EXERCISES.

61. To construct a triangle equivalent to a given square, having given its base and the median drawn from the vertex to the base.

(Draw a ∥ to the base at a distance equal to the altitude of the △.)

What restriction is there on the values of the given lines ?

62. To construct a rhombus equivalent to a given parallelogram, having one of its diagonals coincident with a diagonal of the parallelogram. (Ex. 60.)

63. To draw through a given point within a parallelogram a straight line dividing it into two equivalent parts. (Ex. 49, p. 178.)

64. To construct a parallelogram equivalent to a given trapezoid, having a side and two adjacent angles coincident with one of the non-parallel sides and the adjacent angles, respectively, of the trapezoid. (Ex. 23, p. 176.)

65. To construct a triangle equivalent to a given triangle, having given two of its sides. (Ex. 57.)

(Let m and n be the given sides, and take m as the base.)

Discuss the solution when the altitude is $< n$. $= n$. $> n$.

66. To construct a right triangle equivalent to a given square, having given its hypotenuse. (Ex. 96, p. 119.)

(Find the altitude as in Ex. 56.)

What restriction is there on the values of the given parts?

67. To construct a right triangle equivalent to a given triangle, having given its hypotenuse.

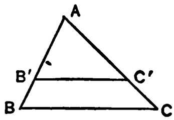
What restriction is there on the values of the given parts?

68. To construct an isosceles triangle equivalent to a given triangle, having given one of its equal sides equal to m.

(Draw a ∥ to the given side at a distance equal to the altitude.)

Discuss the solution when the altitude is $< m$. $= m$. $> m$.

69. To draw a line parallel to the base of a triangle dividing it into two equivalent parts. (§ 319.)



(△ ABC and $AB'C'$ are similar.)

70. To draw through a given point in a side of a parallelogram a straight line dividing it into two equivalent parts.

71. To draw a straight line perpendicular to the bases of a trapezoid, dividing the trapezoid into two equivalent parts.

(A str. line connecting the middle points of the bases divides the trapezoid into two equivalent parts.)

72. To draw through a given point in one of the bases of a trapezoid a straight line dividing the trapezoid into two equivalent parts.

(A str. line connecting the middle points of the bases divides the trapezoid into two equivalent parts.)

73. To construct a triangle similar to two given similar triangles, and equivalent to their sum.

(Construct squares equivalent to the △.)

74. To construct a triangle similar to two given similar triangles, and equivalent to their difference.

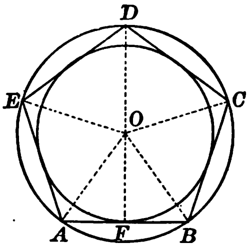
Book V.

REGULAR POLYGONS.—MEASUREMENT OF THE CIRCLE.

339. Def. A *regular polygon* is a polygon which is both equilateral and equiangular.

Prop. I. Theorem.

340. *A circle can be circumscribed about, or inscribed in, any regular polygon.*



Given regular polygon *ABCDE*.

To Prove that a ⊙ can be circumscribed about, or inscribed in, *ABCDE*.

Proof. Let O be the centre of the circumference described through vertices A, B, and C (§ 223).

Draw radii OA, OB, OC, and OD.

In ⊿s OAB and OCD, $OB = OC$. (?)

And since, by def., polygon $ABCDE$ is equilateral,

$$AB = CD.$$

Again, since, by def., polygon *ABCDE* is equiangular,

$$\angle ABC = \angle BCD.$$

And since $\triangle OBC$ is isosceles,

$$\angle OBC = \angle OCB. \qquad (?)$$

$$\therefore \angle ABC - \angle OBC = \angle BCD - \angle OCB.$$

Or, $$\angle OBA = \angle OCD.$$

$$\therefore \triangle OAB = \triangle OCD. \qquad (?)$$

$$\therefore OA = OD. \qquad (?)$$

Then, the circumference which passes through *A*, *B*, and *C* also passes through *D*.

In like manner, it may be proved that the circumference which passes through *B*, *C*, and *D* also passes through *E*.

Hence, a ⊙ can be circumscribed about *ABCDE*.

Again, since *AB*, *BC*, *CD*, etc., are equal chords of the circumscribed ⊙, they are equally distant from *O*. (§ 164)

Hence, a ⊙ described with *O* as a centre, and a line *OF* ⊥ to any side *AB* as a radius, will be inscribed in *ABCDE*.

341. Def. The *centre* of a regular polygon is the common centre of the circumscribed and inscribed circles.

The *angle at the centre* is the angle between the radii drawn to the extremities of any side; as *AOB*.

The *radius* is the radius of the circumscribed circle, *OA*.

The *apothem* is the radius of the inscribed circle, *OF*.

342. Cor. From the equal ≜ *OAB*, *OBC*, etc., we have

$$\angle AOB = \angle BOC = \angle COD, \text{ etc.} \qquad (?)$$

But the sum of these ∠s is four rt. ∠s. (§ 35)

Whence, *the angle at the centre of a regular polygon is equal to four right angles divided by the number of sides.*

EXERCISES.

Find the angle, and the angle at the centre,

1. Of a regular pentagon.

2. Of a regular dodecagon.

3. Of a regular polygon of 32 sides.

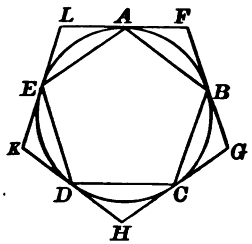
4. Of a regular polygon of 25 sides.

PROP. II. THEOREM.

343. *If the circumference of a circle be divided into any number of equal arcs,*

I. *Their chords form a regular inscribed polygon.*

II. *Tangents at the points of division form a regular circumscribed polygon.*



Given circumference ACD divided into five equal arcs, AB, BC, CD, etc., and chords AB, BC, etc.

Also, lines LF, FG, etc., tangent to $\odot\ ACD$ at A, B, etc., respectively, forming polygon $FGHKL$.

To Prove polygons $ABCDE$ and $FGHKL$ regular.

Proof. Chord $AB =$ chord $BC =$ chord CD, etc. (§ 158)

Again, arc $BCDE =$ arc $CDEA =$ arc $DEAB$, etc., for each is the sum of three of the equal arcs AB, BC, etc.

$\therefore \angle EAB = \angle ABC = \angle BCD$, etc. (§ 193)

Therefore, polygon $ABCDE$ is regular. (§ 339)

Again, in $\triangle ABF$, BCG, CDH, etc., we have

$$AB = BC = CD, \text{ etc.}$$

Also, since arc $AB =$ arc $BC =$ arc CD, etc., we have

$\angle BAF = \angle ABF = \angle CBG = \angle BCG$, etc. (§ 197)

Whence, ABF, BCG, etc., are equal isosceles $\triangle$. (§§ 68, 96)

$$\therefore \angle F = \angle G = \angle H, \text{ etc.},$$

and $$BF = BG = CG = CH, \text{ etc.} \qquad (\S\ 66)$$

$$\therefore FG = GH = HK, \text{ etc.}$$

Therefore, polygon $FGHKL$ is regular. (?)

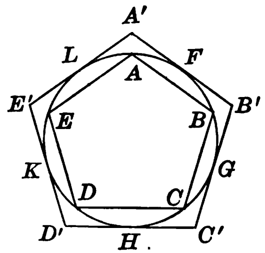
344. Cor. I. 1. *If from the middle point of each arc subtended by a side of a regular inscribed polygon lines be drawn to its extremities, a regular inscribed polygon of double the number of sides is formed.*

2. *If at the middle point of each arc included between two consecutive points of contact of a regular circumscribed polygon tangents be drawn, a regular circumscribed polygon of double the number of sides is formed.*

345. Cor. II. *An equilateral polygon inscribed in a circle is regular;* for its sides subtend equal arcs. (?)

PROP. III. THEOREM.

346. *Tangents to a circle at the middle points of the arcs subtended by the sides of a regular inscribed polygon, form a regular circumscribed polygon.*



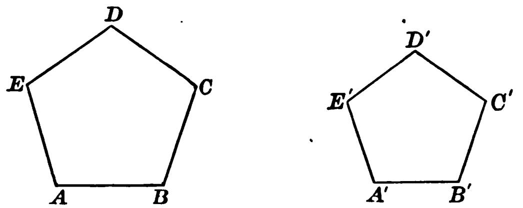
Given $ABCDE$ a regular polygon inscribed in $\odot AC$, and $A'B'C'D'E'$ a polygon whose sides $A'B'$, $B'C'$, etc., are tangent to $\odot AC$ at the middle points F, G, etc., of arcs AB, BC, etc., respectively.

To Prove $A'B'C'D'E'$ a regular polygon.

(Arc AF = arc BF = arc BG = arc CG, etc., and the proposition follows by § 343, II.)

PROP. IV. THEOREM.

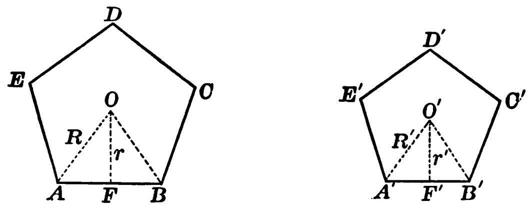
347. *Regular polygons of the same number of sides are similar.*



(The polygons fulfil the conditions of similarity given in § 252.)

PROP. V. THEOREM.

348. *The perimeters of two regular polygons of the same number of sides are to each other as their radii, or as their apothems.*



Given P and P' the perimeters, R and R' the radii, and r and r' the apothems, respectively, of regular polygons AC and $A'C'$ of the same number of sides.

To Prove $$\frac{P}{P'} = \frac{R}{R'} = \frac{r}{r'}.$$

Proof. Let O be the centre of polygon AC, and O' of $A'C'$, and draw lines OA, OB, $O'A'$, and $O'B'$.

Also, draw line $OF \perp AB$, and line $O'F' \perp A'B'$.

Then, $OA = R$, $O'A' = R'$, $OF = r$, and $O'F' = r'$.

Now in isosceles $\triangle$ OAB and $O'A'B'$,

$$\angle AOB = \angle A'O'B'. \qquad (\S\ 342)$$

And since $OA = OB$ and $O'A' = O'B'$, we have

$$\frac{OA}{O'A'} = \frac{OB}{O'B'}.$$

Therefore, $\triangle$ OAB and $O'A'B'$ are similar. (§ 261)

$$\therefore \frac{AB}{A'B'} = \frac{R}{R'} = \frac{r}{r'}. \quad (\S\S\ 253,\ \text{II},\ 264)$$

But polygons AC and $A'C'$ are similar. (§ 347)

$$\therefore \frac{P}{P'} = \frac{AB}{A'B'}. \quad (\S\ 268)$$

$$\therefore \frac{P}{P'} = \frac{R}{R'} = \frac{r}{r'}. \quad (?)$$

349. Cor. Let K denote the area of polygon AC, and K' of $A'C'$.

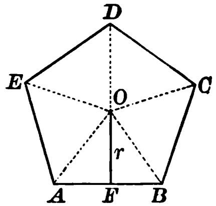
$$\therefore \frac{K}{K'} = \frac{\overline{AB}^2}{\overline{A'B'}^2}. \quad (\S\ 322)$$

But, $\frac{AB}{A'B'} = \frac{R}{R'} = \frac{r}{r'}$; whence, $\frac{K}{K'} = \frac{R^2}{R'^2} = \frac{r^2}{r'^2}$.

That is, *the areas of two regular polygons of the same number of sides are to each other as the squares of their radii, or as the squares of their apothems.*

PROP. VI. THEOREM.

350. *The area of a regular polygon is equal to one-half the product of its perimeter and apothem.*



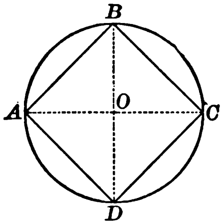
Given the perimeter equal to P, and the apothem OF equal to r, of regular polygon AC.

To Prove area $AC = \frac{1}{2} P \times r$.

($\triangle$ OAB, OBC, etc., have the common altitude r.)

PROP. VII. PROBLEM.

351. *To inscribe a square in a given circle.*



Given $\odot AC$.

Required to inscribe a square in $\odot AC$.

Construction. Draw diameters AC and $BD \perp$ to each other, and chords AB, BC, CD, and DA.

Then, $ABCD$ is an inscribed square.

(The proof is left to the pupil; see § 343, I.)

352. Cor. Denoting radius OA by R, we have

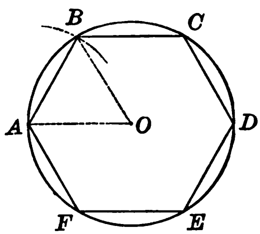
$$\overline{AB}^2 = \overline{OA}^2 + \overline{OB}^2 = 2R^2. \qquad (\S\ 272)$$

$$\therefore AB = R\sqrt{2}.$$

That is, *the side of an inscribed square is equal to the radius of the circle multiplied by* $\sqrt{2}$.

PROP. VIII. PROBLEM.

353. *To inscribe a regular hexagon in a given circle.*



Given $\odot AC$.

Required to inscribe a regular hexagon in ⊙ AC.

Construction. Draw any radius OA.

With A as a centre, and AO as a radius, describe an arc cutting the given circumference at B, and draw chord AB.

Then, AB is a side of a regular inscribed hexagon.

Hence, to inscribe a regular hexagon in a given ⊙, apply the radius six times as a chord.

Proof. Draw radius OB; then, △ OAB is equilateral. (?)

Therefore, △ OAB is equiangular. (§ 95)

Whence, ∠ AOB is one-third of two rt. ∠s. (?)

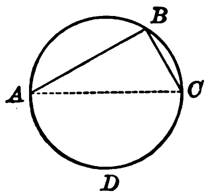
Then, ∠ AOB is one-sixth of four rt. ∠s, and arc AB is one-sixth of the circumference. (§ 154)

Then, AB is a side of a regular inscribed hexagon. (§ 343, I.)

354. Cor. I. *The side of a regular inscribed hexagon is equal to the radius of the circle.*

355. Cor. II. *If chords be drawn joining the alternate vertices of a regular inscribed hexagon, there is formed an inscribed equilateral triangle.*

356. Cor. III. *The side of an inscribed equilateral triangle is equal to the radius of the circle multiplied by $\sqrt{3}$.*



Given AB a side of an equilateral △ inscribed in ⊙ AD whose radius is R.

To Prove $AB = R\sqrt{3}$.

Proof. Draw diameter AC, and chord BC; then, BC is a side of a regular inscribed hexagon. (§ 355)

Now ABC is a rt. △. (§ 195)

$$\therefore \overline{AB}^2 = \overline{AC}^2 - \overline{BC}^2 \quad (?)$$

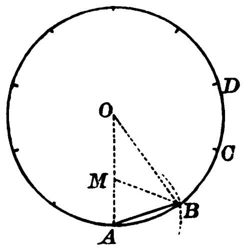
$$= (2R)^2 - R^2 \quad (§ 354)$$

$$= 4R^2 - R^2 = 3R^2.$$

$$\therefore AB = R\sqrt{3}.$$

PROP. IX. PROBLEM.

357. *To inscribe a regular decagon in a given circle.*



Given $\odot AC$.

Required to inscribe a regular decagon in $\odot AC$.

Construction. Draw any radius OA, and divide it internally in extreme and mean ratio at M (§ 297), so that

$$OA : OM = OM : AM. \quad (1)$$

With A as a centre, and OM as a radius, describe an arc cutting the given circumference at B, and draw chord AB.

Then, AB is a side of a regular inscribed decagon.

Hence, to inscribe a regular decagon in a given $\odot$, divide the radius internally in extreme and mean ratio, and apply the greater segment ten times as a chord.

Proof. Draw lines OB and BM.

In $\triangle$ OAB and ABM, $\angle A = \angle A$.

And since, by cons., $OM = AB$, the proportion (1) becomes

$$OA : AB = AB : AM.$$

Therefore, $\triangle$ OAB and ABM are similar. (§ 261)

$$\therefore \angle ABM = \angle AOB. \quad (?)$$

Again, $\triangle OAB$ is isosceles. (?)

Hence, the similar $\triangle ABM$ is isosceles, and

$$AB = BM = OM. \quad \text{(Ax. 1)}$$

$$\therefore \angle OBM = \angle AOB. \quad (?)$$

$$\therefore \angle ABM + \angle OBM = \angle AOB + \angle AOB.$$

Or, $\angle OBA = 2\angle AOB.$ (2)

But since $\triangle OAB$ is isosceles,

$2\angle OBA + \angle AOB = 180°.$ (§ 84)

Then, by (2), $5\angle AOB = 180°$, or $\angle AOB = 36°.$

Therefore, $\angle AOB$ is one-tenth of four rt. $\angle$s, and AB is a side of a regular inscribed decagon. (?)

358. Cor. I. *If chords be drawn joining the alternate vertices of a regular inscribed decagon, there is formed a regular inscribed pentagon.*

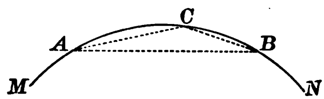
359. Cor. II. Denoting the radius of the ⊙ by R, we have

$$AB = OM = \frac{R(\sqrt{5}-1)}{2}. \qquad (§ 298)$$

This is an expression for the side of a regular inscribed decagon in terms of the radius of the circle.

Prop. X. Problem.

360. *To construct the side of a regular pentedecagon inscribed in a given circle.*



Given arc MN.

Required to construct the side of a regular inscribed polygon of fifteen sides.

Construction. Construct chord AB a side of a regular inscribed hexagon (§ 353), and chord AC a side of a regular inscribed decagon (§ 357), and draw chord BC.

Then, BC is a side of a regular inscribed pentedecagon.

Proof. By cons., arc BC is $\frac{1}{6} - \frac{1}{10}$, or $\frac{1}{15}$, of the circumference.

Hence, chord BC is a side of a regular inscribed pentedecagon. (?)

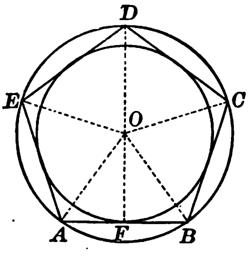
Book V.

REGULAR POLYGONS. — MEASUREMENT OF THE CIRCLE.

339. Def. A *regular polygon* is a polygon which is both equilateral and equiangular.

Prop. I. Theorem.

340. *A circle can be circumscribed about, or inscribed in, any regular polygon.*



Given regular polygon $ABCDE$.

To Prove that a ⊙ can be circumscribed about, or inscribed in, $ABCDE$.

Proof. Let O be the centre of the circumference described through vertices A, B, and C (§ 223).

Draw radii OA, OB, OC, and OD.

In ⊿s OAB and OCD, $OB = OC$. (?)

And since, by def., polygon $ABCDE$ is equilateral,

$$AB = CD.$$

Again, since, by def., polygon $ABCDE$ is equiangular,

$$\angle ABC = \angle BCD.$$

And since $\triangle OBC$ is isosceles,

$$\angle OBC = \angle OCB. \qquad (?)$$

$$\therefore \angle ABC - \angle OBC = \angle BCD - \angle OCB.$$

Or, $$\angle OBA = \angle OCD.$$

$$\therefore \triangle OAB = \triangle OCD. \qquad (?)$$

$$\therefore OA = OD. \qquad (?)$$

Then, the circumference which passes through A, B, and C also passes through D.

In like manner, it may be proved that the circumference which passes through B, C, and D also passes through E.

Hence, a ⊙ can be circumscribed about $ABCDE$.

Again, since AB, BC, CD, etc., are equal chords of the circumscribed ⊙, they are equally distant from O. (§ 164)

Hence, a ⊙ described with O as a centre, and a line OF ⊥ to any side AB as a radius, will be inscribed in $ABCDE$.

341. Def. The *centre* of a regular polygon is the common centre of the circumscribed and inscribed circles.

The *angle at the centre* is the angle between the radii drawn to the extremities of any side; as AOB.

The *radius* is the radius of the circumscribed circle, OA.

The *apothem* is the radius of the inscribed circle, OF.

342. Cor. From the equal ⨻ OAB, OBC, etc., we have

$$\angle AOB = \angle BOC = \angle COD, \text{ etc.} \qquad (?)$$

But the sum of these ⦣ is four rt. ⦣. (§ 35)

Whence, *the angle at the centre of a regular polygon is equal to four right angles divided by the number of sides.*

EXERCISES.

Find the angle, and the angle at the centre,

1. Of a regular pentagon.

2. Of a regular dodecagon.

3. Of a regular polygon of 32 sides.

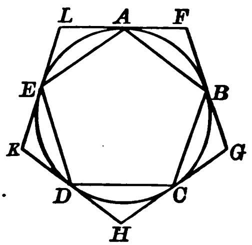
4. Of a regular polygon of 25 sides.

Prop. II. Theorem.

343. *If the circumference of a circle be divided into any number of equal arcs,*

I. *Their chords form a regular inscribed polygon.*

II. *Tangents at the points of division form a regular circumscribed polygon.*



Given circumference ACD divided into five equal arcs, AB, BC, CD, etc., and chords AB, BC, etc.

Also, lines LF, FG, etc., tangent to ⊙ ACD at A, B, etc., respectively, forming polygon $FGHKL$.

To Prove polygons $ABCDE$ and $FGHKL$ regular.

Proof. Chord $AB =$ chord $BC =$ chord CD, etc. (§ 158)

Again, arc $BCDE =$ arc $CDEA =$ arc $DEAB$, etc., for each is the sum of three of the equal arcs AB, BC, etc.

$\therefore \angle EAB = \angle ABC = \angle BCD$, etc. (§ 193)

Therefore, polygon $ABCDE$ is regular. (§ 339)

Again, in △ ABF, BCG, CDH, etc., we have

$$AB = BC = CD, \text{ etc.}$$

Also, since arc $AB =$ arc $BC =$ arc CD, etc., we have

$$\angle BAF = \angle ABF = \angle CBG = \angle BCG, \text{ etc.} \quad (§ 197)$$

Whence, ABF, BCG, etc., are equal isosceles △. (§§ 68, 96)

$\therefore\ \angle F = \angle G = \angle H$, etc.,

and $BF = BG = CG = CH$, etc. (§ 66)

$\therefore\ FG = GH = HK$, etc.

Therefore, polygon $FGHKL$ is regular. (?)

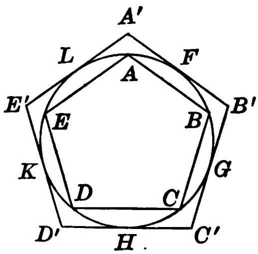
344. Cor. I. 1. *If from the middle point of each arc subtended by a side of a regular inscribed polygon lines be drawn to its extremities, a regular inscribed polygon of double the number of sides is formed.*

2. *If at the middle point of each arc included between two consecutive points of contact of a regular circumscribed polygon tangents be drawn, a regular circumscribed polygon of double the number of sides is formed.*

345. Cor. II. *An equilateral polygon inscribed in a circle is regular;* for its sides subtend equal arcs. (?)

Prop. III. Theorem.

346. *Tangents to a circle at the middle points of the arcs subtended by the sides of a regular inscribed polygon, form a regular circumscribed polygon.*



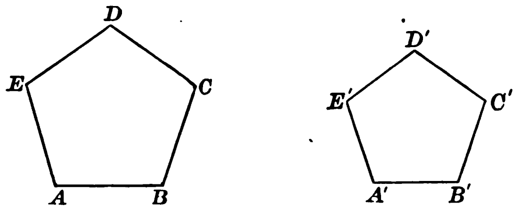
Given $ABCDE$ a regular polygon inscribed in $\odot AC$, and $A'B'C'D'E'$ a polygon whose sides $A'B'$, $B'C'$, etc., are tangent to $\odot AC$ at the middle points F, G, etc., of arcs AB, BC, etc., respectively.

To Prove $A'B'C'D'E'$ a regular polygon.

(Arc AF = arc BF = arc BG = arc CG, etc., and the proposition follows by § 343, II.)

PROP. IV. THEOREM.

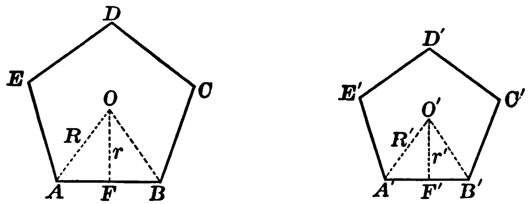
347. *Regular polygons of the same number of sides are similar.*



(The polygons fulfil the conditions of similarity given in § 252.)

PROP. V. THEOREM.

348. *The perimeters of two regular polygons of the same number of sides are to each other as their radii, or as their apothems.*



Given P and P' the perimeters, R and R' the radii, and r and r' the apothems, respectively, of regular polygons AC and $A'C'$ of the same number of sides.

To Prove $$\frac{P}{P'}=\frac{R}{R'}=\frac{r}{r'}.$$

Proof. Let O be the centre of polygon AC, and O' of $A'C'$, and draw lines OA, OB, $O'A'$, and $O'B'$.

Also, draw line $OF \perp AB$, and line $O'F' \perp A'B'$.

Then, $OA = R$, $O'A' = R'$, $OF = r$, and $O'F' = r'$.

Now in isosceles △ OAB and $O'A'B'$,

$$\angle AOB = \angle A'O'B'. \qquad (\S\ 342)$$

And since $OA = OB$ and $O'A' = O'B'$, we have

$$\frac{OA}{O'A'} = \frac{OB}{O'B'}.$$

Therefore, $\triangle$ OAB and $O'A'B'$ are similar. (§ 261)

$$\therefore \frac{AB}{A'B'} = \frac{R}{R'} = \frac{r}{r'}. \quad (§§ 253, II, 264)$$

But polygons AC and $A'C'$ are similar. (§ 347)

$$\therefore \frac{P}{P'} = \frac{AB}{A'B'}. \quad (§ 268)$$

$$\therefore \frac{P}{P'} = \frac{R}{R'} = \frac{r}{r'}. \quad (?)$$

349. Cor. Let K denote the area of polygon AC, and K' of $A'C'$.

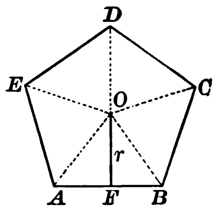
$$\therefore \frac{K}{K'} = \frac{\overline{AB}^2}{\overline{A'B'}^2}. \quad (§ 322)$$

But, $\frac{AB}{A'B'} = \frac{R}{R'} = \frac{r}{r'}$; whence, $\frac{K}{K'} = \frac{R^2}{R'^2} = \frac{r^2}{r'^2}$.

That is, *the areas of two regular polygons of the same number of sides are to each other as the squares of their radii, or as the squares of their apothems.*

PROP. VI. THEOREM.

350. *The area of a regular polygon is equal to one-half the product of its perimeter and apothem.*



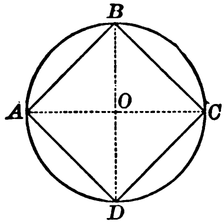
Given the perimeter equal to P, and the apothem OF equal to r, of regular polygon AC.

To Prove area $AC = \frac{1}{2} P \times r$.

($\triangle$ OAB, OBC, etc., have the common altitude r.)

PROP. VII. PROBLEM.

351. *To inscribe a square in a given circle.*



Given ⊙ AC.

Required to inscribe a square in ⊙ AC.

Construction. Draw diameters AC and BD ⊥ to each other, and chords AB, BC, CD, and DA.

Then, $ABCD$ is an inscribed square.

(The proof is left to the pupil; see § 343, I.)

352. Cor. Denoting radius OA by R, we have

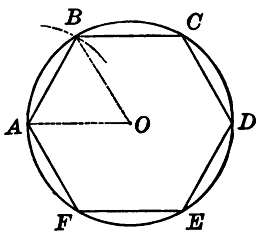
$$\overline{AB}^2 = \overline{OA}^2 + \overline{OB}^2 = 2R^2. \qquad (\S\ 272)$$

$$\therefore AB = R\sqrt{2}.$$

That is, *the side of an inscribed square is equal to the radius of the circle multiplied by* $\sqrt{2}$.

PROP. VIII. PROBLEM.

353. *To inscribe a regular hexagon in a given circle.*



Given ⊙ AC.

Required to inscribe a regular hexagon in ⊙ AC.

Construction. Draw any radius OA.

With A as a centre, and AO as a radius, describe an arc cutting the given circumference at B, and draw chord AB.

Then, AB is a side of a regular inscribed hexagon.

Hence, to inscribe a regular hexagon in a given ⊙, apply the radius six times as a chord.

Proof. Draw radius OB; then, $\triangle\ OAB$ is equilateral. (?)

Therefore, $\triangle\ OAB$ is equiangular. (§ 95)

Whence, $\angle\ AOB$ is one-third of two rt. ∠s. (?)

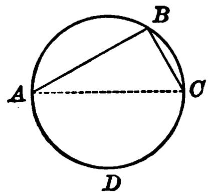
Then, $\angle\ AOB$ is one-sixth of four rt. ∠s, and arc AB is one-sixth of the circumference. (§ 154)

Then, AB is a side of a regular inscribed hexagon. (§ 343, I.)

354. Cor. I. *The side of a regular inscribed hexagon is equal to the radius of the circle.*

355. Cor. II. *If chords be drawn joining the alternate vertices of a regular inscribed hexagon, there is formed an inscribed equilateral triangle.*

356. Cor. III. *The side of an inscribed equilateral triangle is equal to the radius of the circle multiplied by* $\sqrt{3}$.



Given AB a side of an equilateral $\triangle$ inscribed in ⊙ AD whose radius is R.

To Prove $AB = R\sqrt{3}$.

Proof. Draw diameter AC, and chord BC; then, BC is a side of a regular inscribed hexagon. (§ 355)

Now ABC is a rt. $\triangle$. (§ 195)

$$\therefore\ \overline{AB}^2 = \overline{AC}^2 - \overline{BC}^2 \quad (?)$$

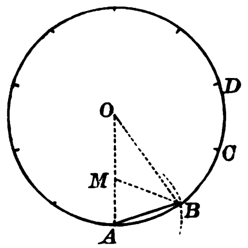
$$= (2R)^2 - R^2 \quad (\S\ 354)$$

$$= 4R^2 - R^2 = 3R^2.$$

$$\therefore\ AB = R\sqrt{3}.$$

PROP. IX. PROBLEM.

357. *To inscribe a regular decagon in a given circle.*



Given $\odot AC$.

Required to inscribe a regular decagon in $\odot AC$.

Construction. Draw any radius OA, and divide it internally in extreme and mean ratio at M (§ 297), so that

$$OA : OM = OM : AM. \quad (1)$$

With A as a centre, and OM as a radius, describe an arc cutting the given circumference at B, and draw chord AB.

Then, AB is a side of a regular inscribed decagon.

Hence, to inscribe a regular decagon in a given $\odot$, divide the radius internally in extreme and mean ratio, and apply the greater segment ten times as a chord.

Proof. Draw lines OB and BM.

In $\triangle$s OAB and ABM, $\angle A = \angle A$.

And since, by cons., $OM = AB$, the proportion (1) becomes

$$OA : AB = AB : AM.$$

Therefore, $\triangle$s OAB and ABM are similar. (§ 261)

$$\therefore \angle ABM = \angle AOB. \quad (?)$$

Again, $\triangle OAB$ is isosceles. (?)

Hence, the similar $\triangle ABM$ is isosceles, and

$$AB = BM = OM. \quad \text{(Ax. 1)}$$

$$\therefore \angle OBM = \angle AOB. \quad (?)$$

$$\therefore \angle ABM + \angle OBM = \angle AOB + \angle AOB.$$

Or, $\angle OBA = 2\angle AOB.$ (2)

But since $\triangle OAB$ is isosceles,

$2\angle OBA + \angle AOB = 180°.$ (§ 84)

Then, by (2), $5\angle AOB = 180°$, or $\angle AOB = 36°.$

Therefore, $\angle AOB$ is one-tenth of four rt. ⦣, and AB is a side of a regular inscribed decagon. (?)

358. Cor. I. *If chords be drawn joining the alternate vertices of a regular inscribed decagon, there is formed a regular inscribed pentagon.*

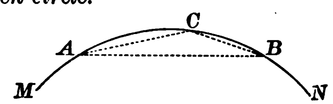
359. Cor. II. Denoting the radius of the ⊙ by R, we have

$$AB = OM = \frac{R(\sqrt{5}-1)}{2}. \qquad (§\ 298)$$

This is an expression for the side of a regular inscribed decagon in terms of the radius of the circle.

Prop. X. Problem.

360. *To construct the side of a regular pentedecagon inscribed in a given circle.*



Given arc MN.

Required to construct the side of a regular inscribed polygon of fifteen sides.

Construction. Construct chord AB a side of a regular inscribed hexagon (§ 353), and chord AC a side of a regular inscribed decagon (§ 357), and draw chord BC.

Then, BC is a side of a regular inscribed pentedecagon.

Proof. By cons., arc BC is $\frac{1}{6} - \frac{1}{10}$, or $\frac{1}{15}$, of the circumference.

Hence, chord BC is a side of a regular inscribed pentedecagon. (?)

361. Sch. I. By bisecting arcs AB, BC, etc., in the figure of Prop. VII., we may construct a regular inscribed octagon (§ 343, I.); and by continuing the bisection, we may construct regular inscribed polygons of 16, 32, 64, etc., sides.

In like manner, by aid of Props. VIII., IX., and X., we may construct regular inscribed polygons of 12, 24, 48, etc., or of 20, 40, 80, etc., or of 30, 60, 120, etc., sides.

362. Sch. II. By drawing tangents to the circumference at the vertices of any one of the above inscribed regular polygons, we may construct a regular circumscribed polygon of the same number of sides. (§ 343, II.)

EXERCISES.

5. The angle at the centre of a regular polygon is the supplement of the angle of the polygon. (§ 127.)

6. The circumference of a circle is greater than the perimeter of any inscribed polygon.

7. An equiangular polygon circumscribed about a circle is regular. (§ 202.)

If r represents the radius, a the apothem, s the side, and k the area,

8. In an equilateral triangle, $a = \frac{1}{2} r$, and $k = \frac{3}{4} r^2 \sqrt{3}$.

9. In a square, $a = \frac{1}{2} r \sqrt{2}$, and $k = 2 r^2$.

10. In a regular hexagon, $a = \frac{1}{2} r \sqrt{3}$, and $k = \frac{3}{2} r^2 \sqrt{3}$.

11. In an equilateral triangle, $r = 2 a$, $s = 2 a \sqrt{3}$, and $k = 3 a^2 \sqrt{3}$.

12. In a square, $r = a \sqrt{2}$, $s = 2 a$, and $k = 4 a^2$.

13. In a regular hexagon, $r = \frac{2}{3} a \sqrt{3}$, and $k = 2 a^2 \sqrt{3}$.

14. In an equilateral triangle, express r, a, and k in terms of s.

15. In a square, express r, a, and k in terms of s.

16. In a regular hexagon, express a and k in terms of s.

17. In an equilateral triangle, express r, a, and s in terms of k.

18. In a square, express r, a, and s in terms of k.

19. In a regular hexagon, express r and a in terms of k.

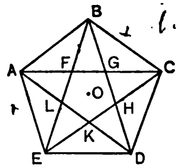
20. The apothem of an equilateral triangle is one-third the altitude of the triangle.

21. The sides of a regular polygon circumscribed about a circle are bisected at the points of contact. (§ 94.)

22. The radius drawn from the centre of a regular polygon to any vertex bisects the angle at that vertex. (§ 44.)

23. The diagonals of a regular pentagon are equal. (§ 63.)

24. The figure bounded by the five diagonals of a regular pentagon is a regular pentagon.



(Prove, by aid of § 164, that a ⊙ can be inscribed in $FGHKL$; then use Ex. 7, p. 198.)

25. The area of a regular inscribed hexagon is a mean proportional between the areas of an inscribed, and of a circumscribed equilateral triangle.

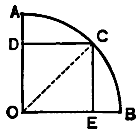
(Prove, by aid of Exs. 8, 10, and 11, p. 198, that the product of the areas of the inscribed and circumscribed equilateral △s is equal to the square of the area of the regular hexagon.)

26. If the diagonals AC and BE of regular pentagon $ABCDE$ intersect at F, prove $BE = AE + AF$. (Ex. 23.)

27. In the figure of Prop. IX., prove that OM is the side of a regular pentagon inscribed in a circle which is circumscribed about triangle OBM.

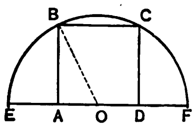
($\angle OBM = 36°$.)

28. The area of the square inscribed in a sector whose central angle is a right angle is equal to one-half the square of the radius.



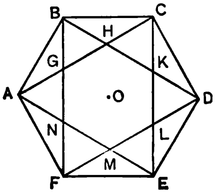
(To prove area $ODCE = \frac{1}{2}\overline{OC}^2$.)

29. The square inscribed in a semicircle is equivalent to two-fifths of the square inscribed in the entire circle.



(By Ex. 9, p. 198, the area of the square inscribed in the entire ⊙ is $2\overline{OB}^2$; we then have to prove area $ABCD = \frac{2}{5}$ of $2\overline{OB}^2 = \frac{4}{5}\overline{OB}^2$.)

30. The diagonals AC, BD, CE, DF, EA, and FB, of regular hexagon $ABCDEF$, form a regular hexagon whose area is equal to one-third the area of $ABCDEF$.



(The apothem of $GHKLMN$ is equal to the apothem of △ACE, which may be found by Ex. 8, p. 198.)

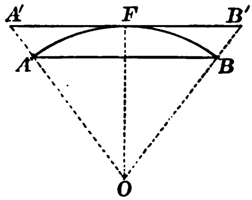
MEASUREMENT OF THE CIRCLE.

PROP. XI. THEOREM.

363. *If a regular polygon be inscribed in, or circumscribed about, a circle, and the number of its sides be indefinitely increased,*

I. *Its perimeter approaches the circumference as a limit.*

II. *Its area approaches the area of the circle as a limit.*



Given p and P the perimeters, and k and K the areas, of two regular polygons of the same number of sides respectively inscribed in, and circumscribed about, a ⊙.

Let C denote the circumference, and S the area, of the ⊙.

I. **To Prove** that, if the number of sides of the polygons be indefinitely increased, P and p approach the limit C.

Proof. Let $A'B'$ be a side of the polygon whose perimeter is P, and draw radius OF to its point of contact.

Also, draw lines OA' and OB' cutting the circumference at A and B, respectively, and chord AB.

Then, AB is a side of the polygon whose perimeter is p. (§ 342)

Now the two polygons are similar. (§ 347)

$$\therefore P:p = OA':OF. \qquad (§ 348)$$

$$\therefore P-p:p = OA'-OF:OF. \qquad (?)$$

$$\therefore (P-p)\times OF = p \times (OA'-OF). \qquad (?)$$

$$\therefore P-p = \frac{p}{OF}\times(OA'-OF).$$

But p is always $<$ the circumference of the $\odot$. (Ax. 4)

Also, $OA' - OF$ is $< A'F$. (§ 62)

$$\therefore\ P - p < \frac{C}{OF} \times A'F. \qquad (1)$$

Now, if the number of sides of each polygon be indefinitely increased, the polygons continuing to have the same number of sides, the length of each side will be indefinitely diminished, and $A'F$ will approach the limit 0.

Then, by (1), since $\frac{C}{OF}$ is a constant, $P - p$ will approach the limit 0.

But the circumference of the $\odot$ is $<$ the perimeter of the circumscribed polygon;* and it is $>$ the perimeter of the inscribed polygon. (Ax. 4)

Then the difference between each perimeter and the circumference, or $P - C$ and $C - p$, will approach the limit 0.

Therefore, P and p will each approach the limit C.

II. **To Prove** that K and k approach the limit S.

Proof. Since the given polygons are similar,

$$K : k = \overline{OA'}^2 : \overline{OF}^2. \qquad (§\ 349)$$

$$\therefore\ K - k : k = \overline{OA'}^2 - \overline{OF}^2 : \overline{OF}^2. \qquad (?)$$

$$\therefore\ (K - k) \times \overline{OF}^2 = k \times (\overline{OA'}^2 - \overline{OF}^2). \qquad (?)$$

$$\therefore\ K - k = \frac{k}{\overline{OF}^2} \times (\overline{OA'}^2 - \overline{OF}^2) = \frac{k}{\overline{OF}^2} \times \overline{A'F}^2. \qquad (?)$$

Now, if the number of sides of each polygon be indefinitely increased, the polygons continuing to have the same number of sides, $A'F$ will approach the limit 0.

Then, $\frac{k}{\overline{OF}^2} \times \overline{A'F}^2$, being always $< \frac{S}{\overline{OF}^2} \times \overline{A'F}^2$, will approach the limit 0.

Whence, $K - k$ will approach the limit 0.

But the area of the $\odot$ is evidently $< K$, and $> k$.

Then, $K - S$ and $S - k$ will each approach the limit 0.

Therefore, K and k will each approach the limit S.

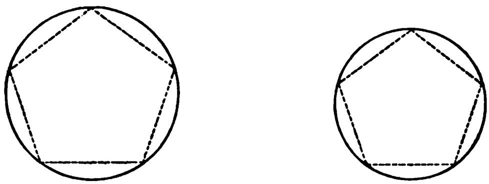
* For a rigorous proof of this statement, see Appendix, p. 386.

364. Cor. 1. *If a regular polygon be inscribed in a circle, and the number of its sides be indefinitely increased, its apothem approaches the radius of the circle as a limit.*

2. *If a regular polygon be circumscribed about a circle, and the number of its sides be indefinitely increased, its radius approaches the radius of the circle as a limit.*

PROP. XII. THEOREM.

365. *The circumferences of two circles are to each other as their radii.*



Given C and C' the circumferences of two ⓢ whose radii are R and R', respectively.

To Prove $$\frac{C}{C'} = \frac{R}{R'}.$$

Proof. Inscribe in the ⓢ regular polygons of the same number of sides; P and P' being the perimeters of the polygons inscribed in ⓢ whose radii are R and R', respectively.

$$\therefore P : P' = R : R'. \qquad (\S\ 348)$$

$$\therefore P \times R' = P' \times R. \qquad (?)$$

Now let the number of sides of each inscribed polygon be indefinitely increased, the two polygons continuing to have the same number of sides.

Then, $P \times R'$ will approach the limit $C \times R'$,

and $P' \times R$ will approach the limit $C' \times R$. (§ 363, I.)

By the Theorem of Limits, these limits are equal. (?)

$$\therefore C \times R' = C' \times R, \text{ or } \frac{C}{C'} = \frac{R}{R'}. \qquad (\S\ 234)$$

366. Cor. I. Multiplying the terms of the ratio $\frac{R}{R'}$ by 2, we have

$$\frac{C}{C'}=\frac{2\,R}{2\,R'}.$$

Now let D and D' denote the diameters of the ⓢ whose radii are R and R', respectively.

$$\therefore\ \frac{C}{C'}=\frac{D}{D'}. \qquad (1)$$

That is, *the circumferences of two circles are to each other as their diameters.*

367. Cor. II. The proportion (1) of § 366 may be written

$$\frac{C}{D}=\frac{C'}{D'}. \qquad (\S\ 235)$$

That is, *the ratio of the circumference of a circle to its diameter has the same value for every circle.*

This constant value is denoted by the symbol π.

$$\therefore\ \frac{C}{D}=\pi. \qquad (1)$$

It is shown by methods of higher mathematics that the ratio π is incommensurable; hence, its numerical value can only be obtained approximately.

Its value to the nearest fourth decimal place is 3.1416.

368. Cor. III. Equation (1) of § 367 gives

$$C=\pi D.$$

That is, *the circumference of a circle is equal to its diameter multiplied by* π.

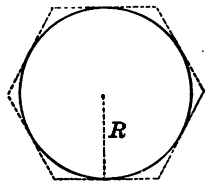
We also have $\qquad C=2\,\pi R.$

That is, *the circumference of a circle is equal to its radius multiplied by* $2\,\pi$.

369. Def. In circles of different radii, *similar arcs, similar segments,* and *similar sectors* are those which correspond to equal central angles.

PROP. XIII. THEOREM.

370. *The area of a circle is equal to one-half the product of its circumference and radius.*



Given R the radius, C the circumference, and S the area, of a ⊙.

To Prove $S = \frac{1}{2} C \times R.$

Proof. Circumscribe a regular polygon about the ⊙.

Let P denote its perimeter, and K its area.

Then since the apothem of the polygon is R,

$$K = \frac{1}{2} P \times R. \qquad (\S\ 350)$$

Now let the number of sides of the circumscribed polygon be indefinitely increased.

Then, K will approach the limit S,

and $\frac{1}{2} P \times R$ will approach the limit $\frac{1}{2} C \times R$. (§ 363)

By the Theorem of Limits, these limits are equal. (?)

$$\therefore\ S = \frac{1}{2} C \times R.$$

371. Cor. I. We have $C = 2\pi R$. (§ 368)

$$\therefore\ S = \pi R \times R = \pi R^2.$$

That is, *the area of a circle is equal to the square of its radius multiplied by* π.

Again, $S = \frac{1}{4}\pi \times 4R^2 = \frac{1}{4}\pi \times (2R)^2.$

Now let D denote the diameter of the ⊙.

$$\therefore\ S = \frac{1}{4}\pi D^2.$$

That is, *the area of a circle is equal to the square of its diameter multiplied by* $\frac{1}{4}\pi$.

372. Cor. II. Let S and S' denote the areas of two ⓢ whose radii are R and R', and diameters D and D', respectively.

$$\therefore \frac{S}{S'} = \frac{\pi R^2}{\pi R'^2} = \frac{R^2}{R'^2},$$

and $$\frac{S}{S'} = \frac{\frac{1}{4}\pi D^2}{\frac{1}{4}\pi D'^2} = \frac{D^2}{D'^2}. \quad (\S\ 371)$$

That is, *the areas of two circles are to each other as the squares of their radii, or as the squares of their diameters.*

373. Cor. III. *The area of a sector is equal to one-half the product of its arc and radius.*

Given s and c the area and arc, respectively, of a sector of a ⊙ whose area, circumference, and radius are S, C, and R, respectively.

To Prove $s = \frac{1}{2} c \times R.$

Proof. A sector is the same part of the ⊙ that its arc is of the circumference.

$$\therefore \frac{s}{S} = \frac{c}{C}, \text{ or } s = c \times \frac{S}{C}.$$

But, $$\frac{S}{C} = \frac{1}{2} R. \quad (\S\ 370)$$

$$\therefore s = \frac{1}{2} c \times R.$$

374. Cor. IV. Since similar sectors are like parts of the ⓢ to which they belong (§ 369), it follows that

Similar sectors are to each other as the squares of their radii.

EXERCISES.

31. Find the circumference and area of a circle whose diameter is 5.

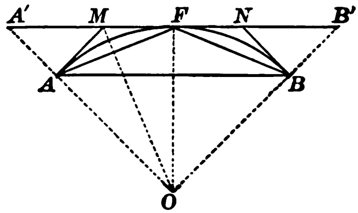
32. Find the radius and area of a circle whose circumference is 25π.

33. Find the diameter and circumference of a circle whose area is 289π.

34. The diameters of two circles are 64 and 88, respectively. What is the ratio of their areas?

PROP. XIV. PROBLEM.

375. *Given p and P, the perimeters of a regular inscribed and of a regular circumscribed polygon of the same number of sides, to find p' and P', the perimeters of a regular inscribed and of a regular circumscribed polygon having double the number of sides.*



Solution. Let AB be a side of the polygon whose perimeter is p, and draw radius OF to middle point of arc AB.

Also, draw radii OA and OB cutting the tangent to the ⊙ at F at points A' and B', respectively; then, $A'B'$ is a side of the polygon whose perimeter is P. (§ 342)

Draw chords AF and BF; also, draw AM and BN tangents to the ⊙ at A and B, meeting $A'B'$ at M and N, respectively.

Then AF and MN are sides of the polygons whose perimeters are p' and P', respectively. (§ 344)

Hence, if n denotes the number of sides of the polygons whose perimeters are p and P, and therefore $2n$ the number of sides of the polygons whose perimeters are p' and P', we have

$$AB = \frac{p}{n},\ A'B' = \frac{P}{n},\ AF = \frac{p'}{2n},\ \text{and}\ MN = \frac{P'}{2n}. \quad (1)$$

Draw line OM; then OM bisects $\angle A'OF$. (§ 175)

$$\therefore\ A'M : MF = OA' : OF. \quad (§ 249)$$

But OA' and OF are the radii of the polygons whose perimeters are P and p, respectively.

$$\therefore\ P : p = OA' : OF. \quad (§ 348)$$

$$\therefore\ P : p = A'M : MF. \qquad (?)$$

$$\therefore\ P + p : p = A'M + MF : MF. \qquad (?)$$

Or, $$\frac{P+p}{p} = \frac{A'F}{MF} = \frac{\frac{1}{2}A'B'}{\frac{1}{2}MN}.$$

Then by (1), $$\frac{P+p}{p} = \frac{\frac{P}{2n}}{\frac{P'}{4n}} = \frac{P}{2n} \times \frac{4n}{P'} = \frac{2P}{P'}.$$

Clearing of fractions,

$$P'(P+p) = 2P \times p.$$

$$\therefore\ P' = \frac{2P \times p}{P+p}. \qquad (2)$$

Again, in isosceles $\triangle$ ABF and AFM,

$$\angle ABF = \angle AFM. \qquad (\S\S\ 193, 197)$$

Therefore, $\triangle$ ABF and AFM are similar. (§ 255)

$$\therefore\ \frac{AF}{AB} = \frac{MF}{AF}. \qquad (?)$$

$$\therefore\ \overline{AF}^2 = AB \times MF. \qquad (?)$$

Then by (1), $$\frac{p'^2}{4n^2} = \frac{p}{n} \times \frac{P'}{4n} = \frac{p \times P'}{4n^2}.$$

$$\therefore\ p'^2 = p \times P'.$$

$$\therefore\ p' = \sqrt{p \times P'}. \qquad (3)$$

PROP. XV. PROBLEM.

376. *To compute an approximate value of* π (§ 367).

Solution. If the diameter of a ⊙ is 1, the side of an inscribed square is $\frac{1}{2}\sqrt{2}$ (§ 352); hence, its perimeter is $2\sqrt{2}$.

Again, the side of a circumscribed square is equal to the diameter of the ⊙; hence, its perimeter is 4.

We then put in equation (2), Prop. XIV.,

$$P = 4, \text{ and } p = 2\sqrt{2} = 2.82843.$$

$$\therefore\ P' = \frac{2P \times p}{P+p} = 3.31371.$$

We then put in equation (3), Prop. XIV.,

$$p = 2.82843, \text{ and } P' = 3.31371.$$

$$\therefore\ p' = \sqrt{p \times P'} = 3.06147.$$

These are the perimeters of the regular circumscribed and inscribed octagons, respectively.

Repeating the operation with these values, we put in (2),

$$P = 3.31371, \text{ and } p = 3.06147.$$

$$\therefore\ P' = \frac{2P \times p}{P+p} = 3.18260.$$

We then put in (3), $p = 3.06147$ and $P' = 3.18260$.

$$\therefore\ p' = \sqrt{p \times P'} = 3.12145.$$

These are, respectively, the perimeters of the regular circumscribed and inscribed polygons of sixteen sides.

In this way, we form the following table:

No. of Sides.	Perimeter of Reg. Circ. Polygon.	Perimeter of Reg. Insc. Polygon.
4	4.	2.82843
8	3.31371	3.06147
16	3.18260	3.12145
32	3.15172	3.13655
64	3.14412	3.14033
128	3.14222	3.14128
256	3.14175	3.14151
512	3.14163	3.14157

The last result shows that the circumference of a ⊙ whose diameter is 1 is $>$ 3.14157, and $<$ 3.14163.

Hence, an approximate value of π is 3.1416, correct to the fourth decimal place.

Note. The value of π to fourteen decimal places is

3.14159265358979.

EXERCISES.

35. The area of a circle is equal to four times the area of the circle described upon its radius as a diameter.

36. The area of one circle is $2\frac{7}{9}$ times the area of another. If the radius of the first is 15, what is the radius of the second?

37. The radii of three circles are 3, 4, and 12, respectively. What is the radius of a circle equivalent to their sum?

38. Find the radius of a circle whose area is one-half the area of a circle whose radius is 9.

39. If the diameter of a circle is 48, what is the length of an arc of $85°$?

40. If the radius of a circle is $3\sqrt{3}$, what is the area of a sector whose central angle is $152°$?

41. If the radius of a circle is 4, what is the area of a segment whose arc is $120°$? $(\pi = 3.1416.)$

(Subtract from the area of the sector whose central $\angle$ is $120°$, the area of the isosceles $\triangle$ whose sides are radii and whose base is the chord of the segment.)

42. Find the area of the circle inscribed in a square whose area is 13.

43. Find the area of the square inscribed in a circle whose area is 196π.

44. If the apothem of a regular hexagon is 6, what is the area of its circumscribed circle?

45. If the length of a quadrant is 1, what is the diameter of the circle? $(\pi = 3.1416.)$

46. The length of the arc subtended by a side of a regular inscribed dodecagon is $\frac{4}{9}\pi$. What is the area of the circle?

47. The perimeter of a regular hexagon circumscribed about a circle is $12\sqrt{3}$. What is the circumference of the circle?

48. The area of a regular hexagon inscribed in a circle is $24\sqrt{3}$. What is the area of the circle?

49. The side of an equilateral triangle is 6. Find the areas of its inscribed and circumscribed circles.

50. The side of a square is 8. Find the circumferences of its inscribed and circumscribed circles.

51. Find the area of a segment having for its chord a side of a regular inscribed hexagon, if the radius of the circle is 10. $(\pi = 3.1416.)$

52. A circular grass-plot, 100 ft. in diameter, is surrounded by a walk 4 ft. wide. Find the area of the walk.

53. Two plots of ground, one a square and the other a circle, contain each 70686 sq. ft. How much longer is the perimeter of the square than the circumference of the circle? ($\pi = 3.1416$.)

54. A wheel revolves 55 times in travelling $\frac{1045\pi}{4}$ ft. What is its diameter in inches?

If r represents the radius, a the apothem, s the side, and k the area, prove that

55. In a regular octagon,

$$s = r\sqrt{2 - \sqrt{2}},\ a = \tfrac{1}{2} r\sqrt{2 + \sqrt{2}},\ \text{and } k = 2r^2\sqrt{2}. \quad (\S\ 375)$$

56. In a regular dodecagon,

$$s = r\sqrt{2 - \sqrt{3}},\ a = \tfrac{1}{2} r\sqrt{2 + \sqrt{3}},\ \text{and } k = 3r^2.$$

57. In a regular octagon,

$$s = 2a(\sqrt{2} - 1),\ r = a\sqrt{4 - 2\sqrt{2}},\ \text{and } k = 8a^2(\sqrt{2} - 1).$$

58. In a regular dodecagon,

$$s = 2a(2 - \sqrt{3}),\ r = 2a\sqrt{2 - \sqrt{3}},\ \text{and } k = 12a^2(2 - \sqrt{3}).$$

59. In a regular decagon, $a = \frac{1}{4} r\sqrt{10 + 2\sqrt{5}}$. (§ 359.)

(Find the apothem by § 273.)

60. What is the number of degrees in an arc whose length is equal to that of the radius of the circle? ($\pi = 3.1416$.)

(Represent the number of degrees by x.)

61. Find the side of a square equivalent to a circle whose diameter is 3. ($\pi = 3.1416$.)

62. Find the radius of a circle equivalent to a square whose side is 10. ($\pi = 3.1416$.)

63. Given one side of a regular hexagon, to construct the hexagon.

64. Given one side of a regular pentagon, to construct the pentagon.

(Draw a ⊙ of any convenient radius, and construct a side of a regular inscribed pentagon.)

65. In a given square, to inscribe a regular octagon.

(Divide the angular magnitude about the centre of the square into eight equal parts.)

66. In a given equilateral triangle to inscribe a regular hexagon.

67. In a given sector whose central angle is a right angle, to inscribe a square.

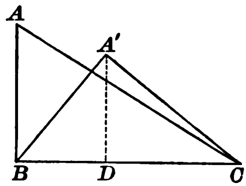
Note. For additional exercises on Book V., see p. 231.

APPENDIX TO PLANE GEOMETRY.

MAXIMA AND MINIMA OF PLANE FIGURES.

PROP. I. THEOREM.

377. *Of all triangles formed with two given sides, that in which these sides are perpendicular is the maximum.*



Given, in $\triangle\!\!\triangle$ ABC and $A'BC$, $AB = A'B$, and $AB \perp BC$.

To Prove area $ABC >$ area $A'BC$.

Proof. Draw $A'D \perp BC$; then,

$$A'B > A'D. \qquad (\S\ 46)$$

$$\therefore\ AB > A'D. \qquad (1)$$

Multiplying both members of (1) by $\frac{1}{2}BC$,

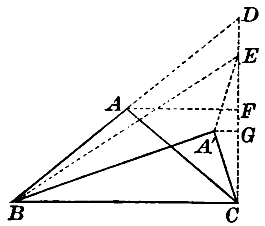
$$\tfrac{1}{2}BC \times AB > \tfrac{1}{2}BC \times A'D.$$

$$\therefore \text{ area } ABC > \text{area } A'BC. \qquad (\S\ 312)$$

378. Def. Two figures are said to be *isoperimetric* when they have equal perimeters.

PROP. II. THEOREM.

379. *Of isoperimetric triangles having the same base, that which is isosceles is the maximum.*



Given ABC and $A'BC$ isoperimetric $\triangle$s, having the same base BC, and $\triangle ABC$ isosceles.

To Prove area $ABC >$ area $A'BC$.

Proof. Produce BA to D, making $AD = AB$, and draw line CD.

Then, $\angle BCD$ is a rt. $\angle$; for it can be inscribed in a semicircle, whose centre is A and radius AB. (§ 195)

Draw lines AF and $A'G \perp$ to CD; take point E on CD so that $A'E = A'C$, and draw line BE.

Then since $\triangle$s ABC and $A'BC$ are isoperimetric,

$$AB + AC = A'B + A'C = A'B + A'E.$$

$$\therefore A'B + A'E = AB + AD = BD.$$

But, $A'B + A'E > BE$. (Ax. 4)

$$\therefore BD > BE.$$

$$\therefore CD > CE. \quad (§ 51)$$

Now AF and $A'G$ are the $\perp$s from the vertices to the bases of isosceles $\triangle$s ACD and $A'CE$, respectively.

$$\therefore CF = \tfrac{1}{2} CD, \text{ and } CG = \tfrac{1}{2} CE. \quad (§ 94)$$

$$\therefore CF > CG. \quad (1)$$

Multiplying both members of (1) by $\tfrac{1}{2} BC$,

$$\tfrac{1}{2} BC \times CF > \tfrac{1}{2} BC \times CG.$$

$$\therefore \text{area } ABC > \text{area } A'BC. \quad (?)$$

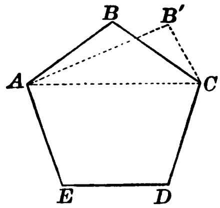
380. Cor. *Of isoperimetric triangles, that which is equilateral is the maximum.*

For if the maximum △ is not isosceles when any side is taken as the base, its area can be increased by making it isosceles. (§ 379)

Then, the maximum △ is equilateral.

PROP. III. THEOREM.

381. *Of isoperimetric polygons having the same number of sides, that which is equilateral is the maximum.*



Given $ABCDE$ the maximum of polygons having the given perimeter and the given number of sides.

To Prove $ABCDE$ equilateral.

Proof. If possible, let sides AB and BC be unequal.

Let $AB'C$ be an isosceles △ with the base AC, having its perimeter equal to that of △ ABC.

$\therefore$ area $AB'C >$ area ABC. (§ 379)

Adding area $ACDE$ to both members,

area $AB'CDE >$ area $ABCDE$.

But this is impossible; for, by hyp., $ABCDE$ is the maximum of polygons having the given perimeter.

Hence, AB and BC cannot be unequal.

In like manner we have

$$BC = CD = DE, \text{ etc.}$$

Then, $ABCDE$ is equilateral.

PROP. IV. THEOREM.

382. *Of isoperimetric equilateral polygons having the same number of sides, that which is equiangular is the maximum.*

Given AB, BC, and CD any three consecutive sides of the maximum of isoperimetric equilateral polygons having the same number of sides.

To Prove $\angle ABC = \angle BCD$.

Proof. There may be three cases:

1. $ABC + BCD = 180°$. (Fig. 1.)
2. $ABC + BCD > 180°$. (Fig. 2.)
3. $ABC + BCD < 180°$. (Fig. 3.)

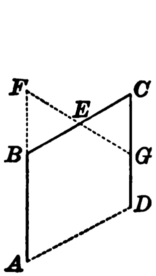


Fig. 1.

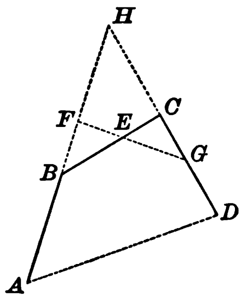


Fig. 2.

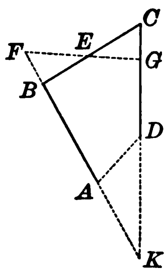


Fig. 3.

If possible, let $\angle ABC$ be $> \angle BCD$, and draw line AD.

In Fig. 1.

Let E be the middle point of BC; and draw line EF, meeting AB produced at F, making $EF = BE$.

Produce FE to meet CD at G.

Then in $\triangle$ BEF and CEG, by hyp., $BE = CE$.

Also, $\angle BEF = \angle CEG$. (?)

And, $\angle EBF = \angle C$,

for each is the supplement of $\angle B$. (§ 33, 2)

$\therefore \triangle BEF = \triangle CEG$. (§§ 86, 68)

$\therefore BE = EF = CE = EG$, and $BF = CG$. (§ 66)

In Fig. 2.

Produce AB and DC to meet at H.

Since, by hyp., $\angle ABC > \angle BCD$, $\angle CBH < \angle BCH$.

$\therefore BH > CH$. (§ 99)

Lay off, on BH, $FH = CH$; and on DH, $GH = BH$; and draw line FG cutting BC at E.

$\therefore \triangle FGH = \triangle BCH$. (§ 63)

$\therefore \angle CBH = \angle FGH$. (§ 66)

Then, in $\triangle$s BEF and CEG, $\angle EBF = \angle CGE$.

Also, $\angle BEF = \angle CEG$. (?)

And $BF = CG$,

since $BF = BH - FH$, and $CG = GH - CH$.

$\therefore \triangle BEF = \triangle CEG$. (§§ 86, 68)

$\therefore BE = CE$ and $EF = EG$. (§ 66)

In Fig. 3.

Produce BA and CD to meet at K.

Since, by hyp., $\angle ABC > \angle BCD$, $CK > BK$. (?)

Lay off, on KB produced, $FK = CK$; and on CK, $GK = BK$; and draw line FG cutting BC at E.

$\therefore \triangle BCK = \triangle FGK$. (?)

$\therefore \angle F = \angle C$. (?)

Then, in $\triangle$s BEF and CEG, $\angle F = \angle C$.

Also, $\angle BEF = \angle CEG$. (?)

And $BF = CG$,

since $BF = FK - BK$, and $CG = CK - GK$.

$\therefore \triangle BEF = \triangle CEG$. (?)

$\therefore BE = CE$ and $EF = EG$. (?)

Then since, in either figure, $BC + CG = BF + FG$, and $\triangle BEF = \triangle CEG$, quadrilateral $AFGD$ is isoperimetric with, and $\approx$ to, quadrilateral $ABCD$.

Calling the remainder of the given polygon P, it follows that the polygon composed of $AFGD$ and P is isoperimetric with, and $\approx$ to, the polygon composed of $ABCD$ and P; that is, the *given* polygon.

Then the polygon composed of $AFGD$ and P must be the maximum of polygons having the given perimeter and the given number of sides.

Hence, the polygon composed of $AFGD$ and P is equilateral. (§ 381)

But this is impossible, since AF is $> DG$.

Hence, $\angle ABC$ cannot be $> \angle BCD$.

In like manner, $\angle ABC$ cannot be $< \angle BCD$.

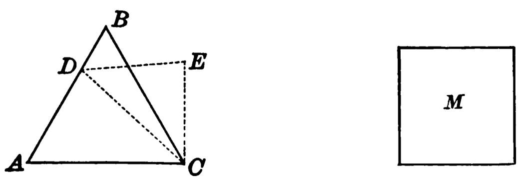
$$\therefore \angle ABC = \angle BCD.$$

Note. The case of *triangles* was considered in § 380. Fig. 3 also provides for the case of triangles by supposing D and K to coincide with A. In the case of *quadrilaterals*, $P = 0$.

383. Cor. *Of isoperimetric polygons having the same number of sides, that which is regular is the maximum.*

Prop. V. Theorem.

384. *Of two isoperimetric regular polygons, that which has the greater number of sides has the greater area.*



Given ABC an equilateral $\triangle$, and M an isoperimetric square.

To Prove area $M >$ area ABC.

Proof. Let D be any point in side AB of $\triangle ABC$.

Draw line DC; and construct isosceles $\triangle CDE$ isoperimetric with $\triangle BCD$, CD being its base.

$$\therefore \text{ area } CDE > \text{ area } BCD. \quad (\S\ 379)$$

$$\therefore \text{ area } ADEC > \text{ area } ABC.$$

But, since $ADEC$ and M are isoperimetric,

$$\text{area } M > \text{ area } ADEC. \quad (\S\ 381)$$

$$\therefore \text{ area } M > \text{ area } ABC.$$

In like manner, we may prove the area of a regular pentagon greater than that of an isoperimetric square; etc.

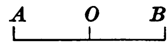
385. Cor. *The area of a circle is greater than the area of any polygon having an equal perimeter.*

SYMMETRICAL FIGURES.

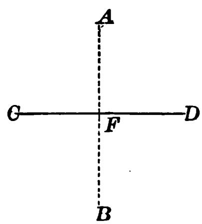
DEFINITIONS.

386. Two points are said to be *symmetrical* with respect to a third, called the *centre of symmetry*, when the latter bisects the straight line which joins them.

Thus, if O is the middle point of straight line AB, points A and B are symmetrical with respect to O as a centre.

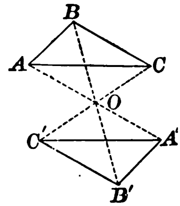


387. Two points are said to be *symmetrical* with respect to a straight line, called the *axis of symmetry*, when the latter bisects at right angles the straight line which joins them.



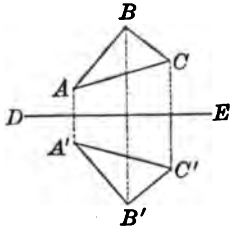
Thus, if line CD bisects line AB at right angles, points A and B are symmetrical with respect to CD as an axis.

388. Two figures are said to be symmetrical with respect to a centre, or with respect to an axis, when to every point of one there corresponds a symmetrical point in the other.



389. Thus, if to every point of triangle *ABC* there corresponds a symmetrical point of triangle *A'B'C'*, with respect to centre *O*, triangle *A'B'C'* is symmetrical to triangle *ABC* with respect to centre *O*.

Again, if to every point of triangle *ABC* there corresponds a symmetrical point of triangle *A'B'C'*, with respect to axis *DE*, triangle *A'B'C'* is symmetrical to triangle *ABC* with respect to axis *DE*.

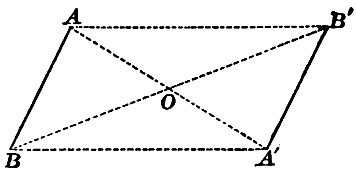


390. A figure is said to be symmetrical with respect to a centre when every straight line drawn through the centre cuts the figure in two points which are symmetrical with respect to that centre.

391. A figure is said to be symmetrical with respect to an axis when it divides it into two figures which are symmetrical with respect to that axis.

Prop. VI. Theorem.

392. *Two straight lines which are symmetrical with respect to a centre are equal and parallel.*



Given str. lines *AB* and *A'B'* symmetrical with respect to centre *O*.

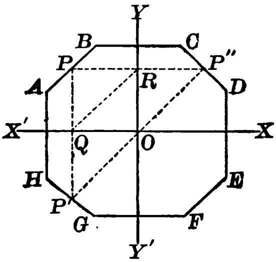
To Prove *AB* and *A'B'* equal and ∥.

Proof. Draw lines *AA'*, *BB'*, *AB'*, and *A'B*.

Then, O bisects AA' and BB'. (§ 386)
Therefore, $AB'A'B$ is a ▱. (§ 112)
Whence, AB and $A'B'$ are equal and ∥. (?)

PROP. VII. THEOREM.

393. *If a figure is symmetrical with respect to two axes at right angles to each other, it is symmetrical with respect to their intersection as a centre.*



Given figure AE symmetrical with respect to axes XX' and YY', intersecting each other at rt. ∠s at O.

To Prove AE symmetrical with respect to O as a centre.

Proof. Let P be any point in the perimeter of AE.

Draw line $PQ \perp XX'$, and line $PR \perp YY'$.

Produce PQ and PR to meet the perimeter of AE at P' and P'', respectively, and draw lines QR, OP', and OP''.

Then since AE is symmetrical with respect to XX',

$$PQ = P'Q. \quad (§ 387)$$

But $PQ = OR$; whence, OR is equal and ∥ to $P'Q$.

Therefore, $OP'QR$ is a ▱. (?)

Whence, QR is equal and ∥ to OP'. (?)

In like manner, we may prove $OP''RQ$ a ▱; and therefore QR equal and ∥ to OP''.

Then since both OP' and OP'' are equal and ∥ to QR, $P'OP''$ is a str. line which is bisected at O.

That is, every str. line drawn through O is bisected at that point, and hence AE is symmetrical with respect to O as a centre. (§ 390)

ADDITIONAL EXERCISES.

BOOK I.

1. Every point within an angle, and not in the bisector, is unequally distant from the sides of the angle.

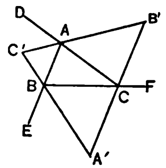
(Prove by *Reductio ad Absurdum.*)

2. If two lines are cut by a third, and the sum of the interior angles on the same side of the transversal is less than two right angles, the lines will meet if sufficiently produced.

(Prove by *Reductio ad Absurdum.*)

3. State and prove the converse of Prop. XXXVII., II.

(Prove $\angle BAD + \angle B = 180°$.)



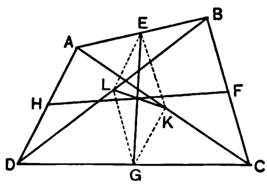
4. The bisectors of the exterior angles of a triangle form a triangle whose angles are respectively the half-sums of the angles of the given triangle taken two and two. (Ex. 69, p. 67.)

(To prove $\angle A' = \frac{1}{2}(\angle ABC + \angle BCA)$, etc.)

5. If CD is the perpendicular from C to side AB of triangle ABC, and CE the bisector of angle C, prove $\angle DCE$ equal to one-half the difference of angles A and B.

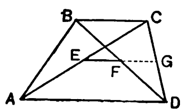
6. If E, F, G, and H are the middle points of sides AB, BC, CD, and DA, respectively, of quadrilateral $ABCD$, prove $EFGH$ a parallelogram whose perimeter is equal to the sum of the diagonals of the quadrilateral. (§ 130.)

7. The lines joining the middle points of the opposite sides of a quadrilateral bisect each other. (Ex. 6, p. 220.)



8. The lines joining the middle points of the opposite sides of a quadrilateral bisect the line joining the middle points of the diagonals.

($EKGL$ is a ▱, and its diagonals bisect each other.)

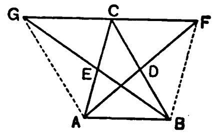


9. The line joining the middle points of the diagonals of a trapezoid is parallel to the bases and equal to one-half their difference.

10. If D is any point in side AC of triangle ABC, and E, F, G, and H the middle points of AD, CD, BC, and AB, respectively, prove $EFGH$ a parallelogram.

11. If E and G are the middle points of sides AB and CD, respectively, of quadrilateral $ABCD$, and K and L the middle points of diagonals AC and BD, respectively, prove $\triangle EKL = \triangle GKL$.

12. If D and E are the middle points of sides BC and AC, respectively, of triangle ABC, and AD be produced to F and BE to G, making $DF = AD$ and $EG = BE$, prove that line FG passes through C, and is bisected at that point.



13. If D is the middle point of side BC of triangle ABC, prove $AD < \frac{1}{2}(AB + AC)$.

(Produce AD to E, making $DE = AD$.)

14. The sum of the medians of a triangle is less than the perimeter, and greater than the semi-perimeter of the triangle.

(Ex. 13, p. 221, and Ex. 106, p. 71.)

15. If the bisectors of the interior angle at C and the exterior angle at B of triangle ABC meet at D, prove $\angle BDC = \frac{1}{2}\angle A$.

16. If AD and BD are the bisectors of the exterior angles at the extremities of the hypotenuse of right triangle ABC, and DE and DF are drawn perpendicular, respectively, to CA and CB produced, prove $CEDF$ a square.

(D is equally distant from AC and BC.)

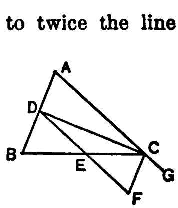
17. AD and BE are drawn from two of the vertices of triangle ABC to the opposite sides, making $\angle BAD = \angle ABE$; if $AD = BE$, prove the triangle isosceles.

18. If perpendiculars AE, BF, CG, and DH, be drawn from the vertices of parallelogram $ABCD$ to any line in its plane, not intersecting its surface, prove

$$AE + CG = BF + DH.$$

(The sum of the bases of a trapezoid is equal to twice the line joining the middle points of the non-parallel sides.)

19. If CD is the bisector of angle C of triangle ABC, and DF be drawn parallel to AC meeting BC at E and the bisector of the angle exterior to C at F, prove $DE = EF$.



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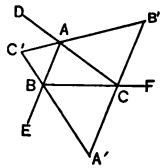

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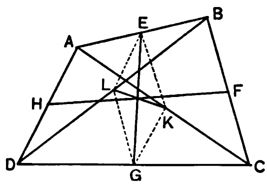

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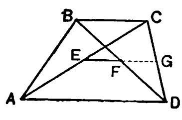
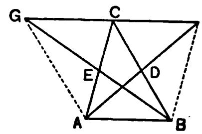


10. If D is any point in side AC of triangle ABC, and E, F, G, and H the middle points of AD, CD, BC, and AB, respectively, prove $EFGH$ a parallelogram.

11. If E and G are the middle points of sides AB and CD, respectively, of quadrilateral $ABCD$, and K and L the middle points of diagonals AC and BD, respectively, prove $\triangle EKL = \triangle GKL$.



12. If D and E are the middle points of sides BC and AC, respectively, of triangle ABC, and AD be produced to F and BE to G, making $DF = AD$ and $EG = BE$, prove that line FG passes through C, and is bisected at that point.

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(D is equally distant from AC and BC.)

17. AD and BE are drawn from two of the vertices of triangle ABC to the opposite sides, making $\angle BAD = \angle ABE$; if $AD = BE$, prove the triangle isosceles.

18. If perpendiculars AE, BF, CG, and DH, be drawn from the vertices of parallelogram $ABCD$ to any line in its plane, not intersecting its surface, prove

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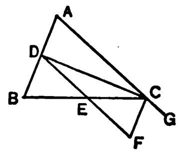


19. If CD is the bisector of angle C of triangle ABC, and DF be drawn parallel to AC meeting BC at E and the bisector of the angle exterior to C at F, prove $DE = EF$.

20. If E and F are the middle points of sides AB and AC, respectively, of triangle ABC, and AD the perpendicular from A to BC, prove $\angle EDF = \angle EAF$. (Ex. 83, p. 69.)

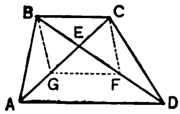
21. If the median drawn from any vertex of a triangle is greater than, equal to, or less than one-half the opposite side, the angle at that vertex is acute, right, or obtuse, respectively. (§ 98.)

22. The number of diagonals of a polygon of n sides is $\frac{n(n-3)}{2}$.

23. The sum of the medians of a triangle is greater than three-fourths the perimeter of the triangle.

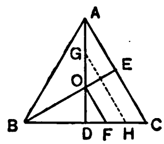
(Fig. of Prop. LII. Since $AO = \frac{2}{3} AD$ and $BO = \frac{2}{3} BE$, we have $AB < \frac{2}{3}(AD + BE)$, by Ax. 4.)

24. If the lower base AD of trapezoid $ABCD$ is double the upper base BC, and the diagonals intersect at E, prove $CE = \frac{1}{3} AC$ and $BE = \frac{1}{3} BD$.



(Let F be the middle point of DE, and G of AE.)

25. If O is the point of intersection of the medians AD and BE of equilateral triangle ABC, and line OF be drawn parallel to side AC, meeting side BC at F, prove that DF is equal to $\frac{1}{6} BC$. (§ 133.)



(Let G be the middle point of OA.)

26. If equiangular triangles be constructed on the sides of a triangle, the lines drawn from their outer vertices to the opposite vertices of the triangle are equal. (§ 63.)

27. If two of the medians of a triangle are equal, the triangle is isosceles.

(Fig. of Prop. LII. Let $AD = BE$.)

BOOK II.

28. AB and AC are the tangents to a circle from point A, and D is any point in the smaller of the arcs subtended by chord BC. If a tangent to the circle at D meets AB at E and AC at F, prove the perimeter of triangle AEF constant. (§ 174.)

29. The line joining the middle points of the arcs subtended by sides AB and AC of an inscribed triangle ABC cuts AB at F and AC at G. Prove $AF = AG$.

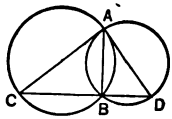
($\angle AFG = \angle AGF$.)

30. If $ABCD$ is a circumscribed quadrilateral, prove the angle between the lines joining the opposite points of contact equal to $\frac{1}{2}(A+C)$. (§ 202.)

31. If sides AB and BC of inscribed hexagon $ABCDEF$ are parallel to sides DE and EF, respectively, prove side AF parallel to side CD. (§ 172.)

(Draw line CF, and prove $\angle AFC = \angle FCD$.)

32. If AB is the common chord of two intersecting circles, and AC and AD diameters drawn from A, prove that line CD passes through B. (§ 195.)



33. If AB is a common exterior tangent to two circles which touch each other externally at C, prove $\angle ACB$ a right angle.

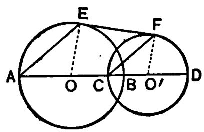
(Draw the common tangent at C, meeting AB at D.)

34. If AB and AC are the tangents to a circle from point A, and D is any point on the greater of the arcs subtended by chord BC, prove the sum of angles ABD and ACD constant.

35. If A, C, B, and D are four points in a straight line, B being between C and D, and EF is a common tangent to the circles described upon AB and CD as diameters, prove

$$\angle BAE = \angle DCF.$$

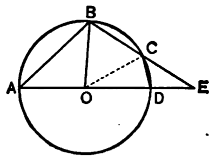
(We have $OE \parallel O'F$.)



36. $ABCD$ is an inscribed quadrilateral, AD being a diameter of the circle. If O is the centre, and sides AD and BC produced meet at E making $CE = OA$, prove

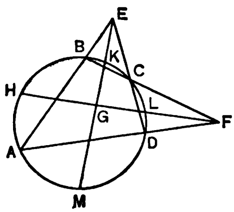
$$\angle AOB = 3\angle CED.$$

($\angle AOB$ is an ext. $\angle$ of $\triangle OBE$, and $\angle BCO$ of $\triangle OCE$.)



37. $ABCD$ is a quadrilateral inscribed in a circle. If sides AB and DC produced intersect at E, and sides AD and BC produced at F, prove the bisectors of angles E and F perpendicular. (§ 199.)

(Prove arc HM + arc $KL = 180°$.)



38. If $ABCD$ is an inscribed quadrilateral, and sides AD and BC produced meet at P, the tangent at P to the circle circumscribed about triangle ABP is parallel to CD. (§ 196.)

(Prove $\angle$ between the tangent and BP equal to $\angle PCD$.)

39. $ABCD$ is a quadrilateral inscribed in a circle. Another circle is described upon AD as a chord, meeting AB and CD at E and F, respectively. Prove chords BC and EF parallel.

(Prove $\angle ABC = \angle AEF$.)

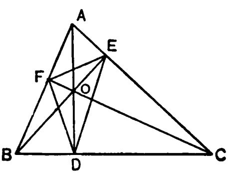
40. If $ABCDEFGH$ is an inscribed octagon, the sum of angles A, C, E, and G is equal to six right angles. (§ 193.)

41. If the number of sides of an inscribed polygon is even, the sum of the alternate angles is equal to as many right angles as the polygon has sides less two.

(Use same method of proof as in Ex. 40.)

42. If a right triangle has for its hypotenuse the side of a square, and lies without the square, the straight line drawn from the centre of the square to the vertex of the right angle bisects the right angle. (§ 200.)

43. The perpendiculars from the vertices of a triangle to the opposite sides are the bisectors of the angles of the triangle formed by joining the feet of the perpendiculars.



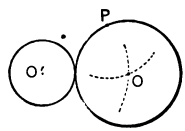
(To prove AD, BE, and CF the bisectors of the $\angle$s of $\triangle DEF$. By § 200, a $\odot$ can be circumscribed about quadrilateral $BDOF$; then $\angle ODF = \angle OBF$; in this way, $\angle ODF = 90° - \angle BAC$.)

Constructions.

44. Given a side, an adjacent angle, and the radius of the circumscribed circle of a triangle, to construct the triangle.

What restriction is there on the values of the given lines?

45. To describe a circle of given radius tangent to a given circle, and passing through a given point without the circle.



46. To draw between two given intersecting lines a straight line which shall be equal to one given straight line, and parallel to another.

(Draw a $\parallel$ to one of the intersecting lines.)

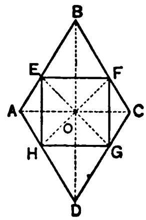
47. Given an angle of a triangle, the length of its bisector, and the length of the perpendicular from its vertex to the opposite side, to construct the triangle.

(The side opposite the given ∠ is tangent to a ⊙ drawn with the vertex as a centre, and with the ⊥ from the vertex to the opposite side as a radius.)

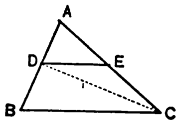
48. Given an angle of a triangle, and the segments of the opposite side made by the perpendicular from its vertex, to construct the triangle. (§ 226.)

49. To inscribe a square in a given rhombus.

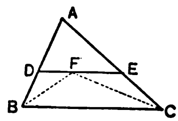
(Bisect the ∠s between diagonals AC and BD. To prove $EFGH$ a square, prove △s OBE, OBF, ODG, and ODH equal; whence, $OE = OF = OG = OH$.)



50. To draw a parallel to side BC of triangle ABC meeting AB and AC in D and E, respectively, so that DE may equal EC.



51. To draw a parallel to side BC of triangle ABC, meeting AB and AC in D and E, respectively, so that DE may equal the sum of BD and CE.



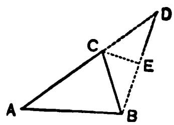
52. Given an angle of a triangle, the length of the perpendicular from the vertex of another angle to the opposite side, and the radius of the circumscribed circle, to construct the triangle.

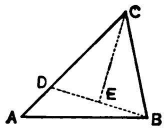
(The centre of the circumscribed ⊙ is equally distant from the given vertices.)

53. Through a given point without a given circle to draw a secant whose internal and external segments shall be equal. (Ex. 65, p. 103.)

54. Given the base of a triangle, an adjacent angle, and the sum of the other two sides, to construct the triangle.

(Lay off AD equal to the sum of the other two sides.)



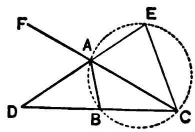


55. Given the base of a triangle, an adjacent acute angle, and the difference of the other two sides, to construct the triangle.

What restriction is there on the values of the given lines?

56. Given the feet of the perpendiculars from the vertices of a triangle to the opposite sides, to construct the triangle. (Ex. 43.)

BOOK III.



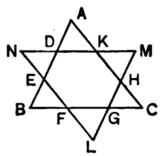
57. In any triangle, the product of any two sides is equal to the product of the segments of the third side formed by the bisector of the exterior angle at the opposite vertex, minus the square of the bisector.

(To prove $AB \times AC = DB \times DC - \overline{AD}^2$. The work is carried out as in § 288; first prove ≜ ABD and ACE similar.)

58. If the sides of a triangle are $AB = 4$, $AC = 5$, and $BC = 6$, find the length of the bisector of the exterior angle at vertex A. (§ 251.)

59. ABC is an isosceles triangle. If the perpendicular to AB at A meets base BC, produced if necessary, at E, and D is the middle point of BE, prove AB a mean proportional between BC and BD. (Ex. 83, p. 69.)

(≜ ABC and ABD are similar.)



60. If D and E, F and G, and H and K are points on sides AB, BC, and CA, respectively, of triangle ABC, so taken that $AD = DE = EB$, $BF = FG = GC$, and $CH = HK = KA$, prove that lines EF, GH, and KD, when produced, form a triangle equal to ABC.

(By § 248, sides of $\triangle LMN$ are $\parallel$, respectively, to sides of $\triangle ABC$.)

61. The square of the common tangent to two circles which are tangent to each other externally is equal to 4 times the product of their radii. (§ 273.)

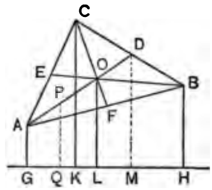
62. The sides AB and BC of triangle ABC are 3 and 7, respectively, and the length of the bisector of the exterior angle B is $3\sqrt{7}$. Find side AC. (Ex. 57, and § 251.)

63. One segment of a chord drawn through a point 7 units from the centre of a circle is 4 units. If the diameter of the circle is 15 units, what is the other segment? (§ 280.)

64. If E is the middle point of one of the parallel sides BC of trapezoid $ABCD$, and AE and DE produced meet DC and AB produced at F and G, respectively, prove FG parallel to AD.

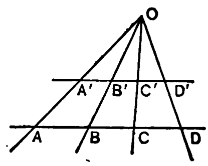
($\triangle$ ADG and BEG are similar, as also are $\triangle$ ADF and CEF.)

65. The perpendicular from the intersection of the medians of a triangle to any straight line in the plane of the triangle, not intersecting its surface, is equal to one-third the sum of the perpendiculars from the vertices of the triangle to the same line.



(The sum of the bases of a trapezoid is equal to twice the line joining the middle points of the non-parallel sides.)

66. If two parallels are cut by three or more straight lines passing through a common point, the corresponding segments are proportional.



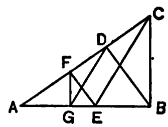
(To prove $\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CD}{C'D'}$. $\triangle$ OAB, OBC, and OCD are similar, respectively, to $\triangle$ $OA'B'$, $OB'C'$, and $OC'A'$.)

67. State and prove the converse of Ex. 66.

(Fig. of Ex. 66. To prove that AA', BB', CC', and DD' pass through a common point. Let AA' and BB' meet at O, and draw OC and OC'; then prove $\triangle$ OBC and $OB'C'$ similar.)

68. The non-parallel sides of a trapezoid and the line joining the middle points of the parallel sides, if produced, meet in a common point. (Ex. 67.)

69. BD is the perpendicular from the vertex of the right angle to the hypotenuse of right triangle ABC. If E is any point in AB, and EF be drawn perpendicular to AC, and FG perpendicular to AB, prove lines CE and DG parallel.



($\triangle$ ABC and AEF are similar. By § 271, 2, we may prove $AD : CD = \overline{AB}^2 : \overline{BC}^2$, and $AG : EG = \overline{AF}^2 : \overline{EF}^2$; then, we have $AD : CD = AG : EG$.)

70. In right triangle ABC, $\overline{BC}^2 = 3\,\overline{AC}^2$. If CD be drawn from the vertex of the right angle to the middle point of AB, prove $\angle ACD$ equal to 60°. (Ex. 83, p. 69.)

(Prove $AC = \frac{1}{2} AB$.)

71. If D is the middle point of side BC of right triangle ABC, and DE be drawn perpendicular to hypotenuse AB, prove

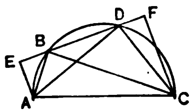
$$\overline{AE}^2 - \overline{BE}^2 = \overline{AC}^2.$$

($AE = AB - BE$; square this by the rule of Algebra.)

72. If BE and CF are medians drawn from the extremities of the hypotenuse of right triangle ABC, prove

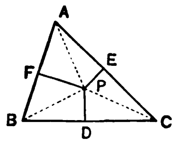
$$4\,\overline{BE}^2 + 4\,\overline{CF}^2 = 5\,\overline{BC}^2. \quad (\S\ 272.)$$

73. If ABC and ADC are angles inscribed in a semicircle, and AE and CF be drawn perpendicular to BD produced, prove



$$\overline{BE}^2 + \overline{BF}^2 = \overline{DE}^2 + \overline{DF}^2. \quad (\S\ 273.)$$

74. If perpendiculars PF, PD, and PE be drawn from any point P to sides AB, BC, and CA, respectively, of a triangle, prove



$$\overline{AF}^2 + \overline{BD}^2 + \overline{CE}^2 = \overline{AE}^2 + \overline{BF}^2 + \overline{CD}^2.$$

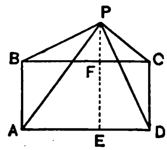
(§ 273.)

75. If BC is the hypotenuse of right triangle ABC, prove

$$(AB + BC + CA)^2 = 2(AB + BC)(BC + CA).$$

(Square $AB+BC+CA$ by the rule of Algebra.)

76. If lines be drawn from any point P to the vertices of rectangle $ABCD$, prove



$$\overline{PA}^2 + \overline{PC}^2 = \overline{PB}^2 + \overline{PD}^2.$$

77. If AB and AC are the equal sides of an isosceles triangle, and BD be drawn perpendicular to AC, prove $2\,AC \times CD = \overline{BC}^2$.

($AD = AC - CD$; square this by the rule of Algebra.)

78. If AD and BE are the perpendiculars from vertices A and B, respectively, of acute-angled triangle ABC to the opposite sides, prove

$$AC \times AE + BC \times BD = \overline{AB}^2.$$

(By § 277, $2\,AC \times AE = \overline{AB}^2 + \overline{AC}^2 - \overline{BC}^2$; and in like manner a value may be found for $2\,BC \times BD$.)

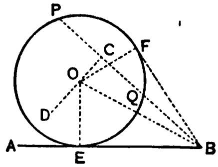
79. The sum of the squares of the sides of a parallelogram is equal to the sum of the squares of its diagonals. (§ 279, I.)

80. To construct a triangle similar to a given triangle, having given its perimeter.

(Divide the perimeter into parts proportional to the sides of the △.)

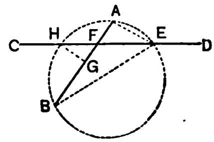
81. To construct a right triangle, having given its perimeter and an acute angle.

(From any point in one side of the given ∠ draw a ⊥ to the other side.)



82. To describe a circle through two given points, tangent to a given straight line. (§ 282.)

(To prove ⊙ draw with O as a centre and OP as a radius tangent to AB, draw BF tangent to the ⊙, and prove △ OBE = △ OBF.)



83. If A and B are points on either side of line CD, and line AB cuts CD at F, find a point E in CD such that

$$AE : BE = AF : BF. \quad (\S\ 249.)$$

(EF bisects ∠ AEB of △ ABE.)

BOOK IV.

84. In the figure on p. 174,

(a) Prove lines CF and BH perpendicular.

(If CF and BH meet at S, ∠ CSH is an ext. ∠ of △ BCS.)

(b) Prove lines AG and BK parallel.

(c) Prove the sum of the perpendiculars from H and L to AB produced equal to AB.

(If ⊥ from H meets BA produced at Q, △ AHQ = △ ACD.)

(d) Prove triangles AFH, BEL, and CGK each equivalent to ABC.

(If AF be taken as the base of △ AFH, its altitude is equal to CD.)

(e) Prove C, H, and L in the same straight line.

(Prove CH and CL in the same str. line.)

(f) Prove the square described upon the sum of AC and BC equivalent to the square described upon AB, plus 4 times △ ABC.

(Square $AC + BC$ by the rule of Algebra.)

(g) Prove the sum of angles AFH, AHF, BEL, and BLE equal to a right angle.

($\angle AFH + \angle AHF = 180^\circ - \angle FAH$.)

(h) If FN and EP are the perpendiculars from F and E, respectivély, to HA and LB produced, prove triangles AFN and BEP each equal to ABC.

(i) Prove $\overline{EL}^2 + \overline{FH}^2 + \overline{GK}^2 = 6\,\overline{AB}^2$.

(EL is the hypotenuse of rt. △ ELP, and FH of △ FHN; sides PL and HN may be found by (h).)

(*j*) Prove $\overline{CF}^2 - \overline{CE}^2 = \overline{AC}^2 - \overline{BC}^2$.

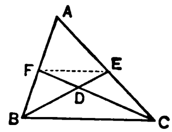
(*k*) Prove that lines *AL*, *BH*, and *CM* meet at a common point. (Ex. 84, (*a*).)

(Produce *DC* to *T*, making $CT = DM$, and prove *AL*, *BH*, and *CM* the ⊥s from the vertices to the opposite sides of $\triangle ABT$.)

(*l*) Prove that lines *HG*, *LK*, and *MC* when produced meet at a common point.

(Draw *GT* and *KT*, and prove △s *CGT* and *CKT* rt. △s.)

85. If *BE* and *CF* are medians drawn from vertices *B* and *C* of triangle *ABC*, intersecting at *D*, prove triangle *BCD* equivalent to quadrilateral *AEDF*.



(area BCD = area BCF − area BDF.)

86. If *D* is the middle point of side *BC* of triangle *ABC*, *E* the middle point of *AD*, *F* of *BE*, and *G* of *CF*, prove $\triangle ABC$ equivalent to $8 \triangle EFG$.

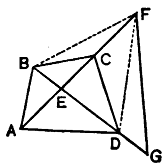
(Draw *CE*; then, area $ABC = 2$ area BCE.)

87. If *E* and *F* are the middle points of sides *AB* and *CD*, respectively, of parallelogram *ABCD*, and *AF* and *CE* be drawn intersecting *BD* in *H* and *L*, respectively, and *BF* and *DE* intersecting *AC* in *K* and *G*, respectively, prove *GHKL* a parallelogram equivalent to $\frac{1}{6}ABCD$. (§ 140.)

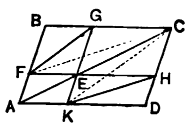
(If *AC* and *BD* intersect at *M*, *AM* and *DE* are medians of $\triangle ABD$.)

88. Any quadrilateral *ABCD* is equivalent to a triangle, two of whose sides are equal to diagonals *AC* and *BD*, respectively, and include an angle equal to either of the angles between *AC* and *BD*.

(Produce *AC* to *F*, making $CF = AE$; and *BD* to *G*, making $DG = BE$. To prove quadrilateral $ABCD \approx \triangle EFG$. $\triangle DFG \approx \triangle ABC$.)



89. If through any point *E* in diagonal *AC* of parallelogram *ABCD* parallels to *AD* and *AB* be drawn, meeting *AB* and *CD* in *F* and *H*, respectively, and *BC* and *AD* in *G* and *K*, respectively, prove triangles *EFG* and *EHK* equivalent.



90. If *E* is the intersection of diagonals *AC* and *BD* of a quadrilateral, and triangles *ABE* and *CDE* are equivalent, prove sides *AD* and *BC* parallel.

(△s *ABD* and *ACD* are equivalent.)

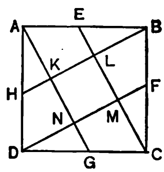
91. Find the area of a trapezoid whose parallel sides are 28 and 36, and non-parallel sides 15 and 17, respectively.

(By drawing through one vertex of the upper base a ∥ to one of the non-parallel sides, one ∠ of the figure may be proved a rt. ∠, by Ex. 63, p. 154.)

92. If similar polygons be described upon the legs of a right triangle as homologous sides, the polygon described upon the hypotenuse is equivalent to the sum of the polygons described upon the legs.

(Find, by § 322, the ratio of the area of the polygon described upon each leg to the area of the polygon described upon the hypotenuse.)

93. If E, F, G, and H are the middle points of sides AB, BC, CD, and DA, respectively, of a square, prove that lines AG, BH, CE, and DF form a square equivalent to $\frac{1}{5}ABCD$.



(First prove $\triangle ADG = \triangle ABH$; then, by § 85, 1, $\angle NKL$ may be proved a rt. ∠. By § 131, each side of $KLMN$ may be proved equal to AK. From similar ⚠ AHK and ADG, AK may be proved equal to $\frac{AD}{\sqrt{5}}$.)

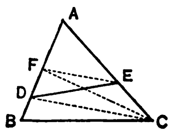
94. If E is any point in side BC of parallelogram $ABCD$, and DE be drawn meeting AB produced at F, prove triangles ABE and CEF equivalent.

($\triangle ABE + \triangle CDE \approx \triangle CDF$.)

95. If D is a point in side AB of triangle ABC, find a point E in AC such that triangle ADE shall be equivalent to one-half triangle ABC.

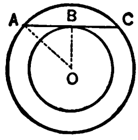
($\triangle DEF \approx \triangle CEF$)

What restriction is there on the position of D?



BOOK V.

96. The area of the ring included between two concentric circles is equal to the area of a circle, whose diameter is that chord of the outer circle which is tangent to the inner.



(To prove area of ring $= \frac{1}{4}\pi\overline{AC}^2$.)

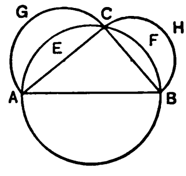
97. An equilateral polygon circumscribed about a circle is regular if the number of its sides is odd. (§ 345.)

(The polygon can be inscribed in a ⊙.)

98. An equiangular polygon inscribed in a circle is regular if the number of its sides is odd. (§ 345.)

(The polygon can be proved equilateral.)

99. If a circle be circumscribed about a right triangle, and on each of its legs as a diameter a semicircle be described exterior to the triangle, the sum of the areas of the crescents thus formed is equal to the area of the triangle. (§ 272.)



(To prove area $AECG$ + area $BFCH$ equal to area ABC.)

100. If the radius of the circle is 1, the side, apothem, and diagonal of a regular inscribed pentagon are, respectively,

$$\frac{1}{2}\sqrt{(10-2\sqrt{5})},\ \frac{1}{4}(1+\sqrt{5}),\ \text{and}\ \frac{1}{2}\sqrt{(10+2\sqrt{5})}.$$

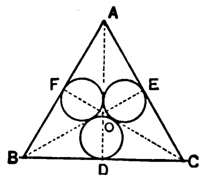
(In Fig. of Prop. IX., the apothem of a regular inscribed pentagon is the distance from O to the foot of a ⊥ from B to OA, and its side is twice this ⊥. The diagonal is a leg of a rt. △ whose hypotenuse is a diameter, and whose other leg is a side of a regular inscribed decagon.)

101. The square of the side of a regular inscribed pentagon, minus the square of the side of a regular inscribed decagon, is equal to the square of the radius. (Ex. 100, and § 359.)

102. The sum of the perpendiculars drawn to the sides of a regular polygon from any point within the figure is equal to the apothem multiplied by the number of sides of the polygon.

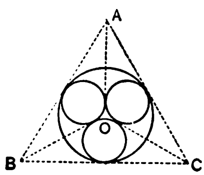
(The ⊥s are the altitudes of △s which make up the polygon.)

103. In a given equilateral triangle to inscribe three equal circles, tangent to each other, and each tangent to one, and only one, side of the triangle.



(By § 174, the ⊙s touch the ⊥s at the same points.)

104. In a given circle to inscribe three equal circles, tangent to each other and to the given circle.



SOLID GEOMETRY.

Book VI.

LINES AND PLANES IN SPACE. DIEDRAL ANGLES. POLYEDRAL ANGLES.

394. Def. A plane is said to be *determined* by certain lines or points when one plane, and only one, can be drawn through these lines or points.

PROP. I. THEOREM.

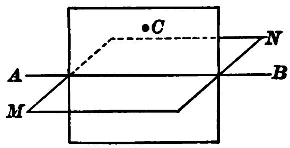
395. *A plane is determined*

I. *By a straight line and a point without the line.*

II. *By three points not in the same straight line.*

III. *By two intersecting straight lines.*

IV. *By two parallel straight lines.*

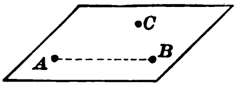


I. **Given** point C without str. line AB.

To Prove that a plane is determined by AB and C.

Proof. If any plane MN be drawn through AB, it may be revolved about AB as an axis until it contains point C.

Hence, a plane can be drawn through line AB and point C; and it is evident that but one such plane can be drawn.



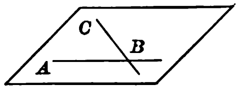
II. **Given** *A*, *B*, and *C* three points not in the same str. line.

To Prove that a plane is determined by *A*, *B*, and *C*.

Proof. Draw line *AB*; then a plane, and only one, can be drawn through line *AB* and point *C*.

[A plane is determined by a str. line and a point without the line.] (§ 395, I)

Then, a plane, and only one, can be drawn through *A*, *B*, and *C*.



III. **Given** *AB* and *BC* intersecting str. lines.

To Prove that a plane is determined by *AB* and *BC*.

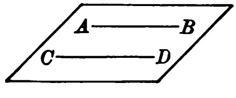
Proof. A plane, and only one, can be drawn through line *AB* and point *C*.

[A plane is determined by a str. line and a point without the line.] (§ 395, I)

And since this plane contains points *B* and *C*, it must contain line *BC*.

[A plane is a surface such that the str. line joining any two of its points lies entirely in the surface.] (§ 9)

Then, a plane, and only one, can be drawn through *AB* and *BC*.



IV. **Given** ‖s *AB* and *CD*.

To Prove that a plane is determined by *AB* and *CD*.

Proof. The ‖s *AB* and *CD* lie in the same plane.

[Two str. lines are said to be ‖ when they lie in the same plane, and cannot meet however far they may be produced.] (§ 52)

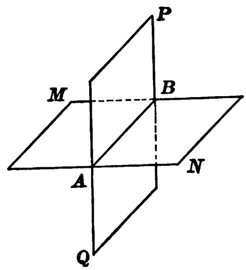
And only one plane can be drawn through AB and point C.

[A plane is determined by a str. line and a point without the line.] (§ 395, I)

Then, a plane, and only one, can be drawn through AB and CD.

PROP. II. THEOREM.

396. *The intersection of two planes is a straight line.*



Given line AB the intersection of planes MN and PQ.

To Prove AB a str. line.

Proof. Draw a str. line between points A and B.

This str. line lies in plane MN, and also in plane PQ.

[A plane is a surface such that the str. line joining any two of its points lies entirely in the surface.] (§ 9)

Then it must be the intersection of planes MN and PQ.

Hence, the line of intersection AB is a str. line.

397. Defs. If a straight line meets a plane, the point of intersection is called the *foot* of the line.

A straight line is said to be *perpendicular to a plane* when it is perpendicular to every straight line drawn in the plane through its foot.

A straight line is said to be *parallel to a plane* when it cannot meet the plane however far they may be produced.

A straight line which is neither perpendicular nor parallel to a plane, is said to be *oblique* to it.

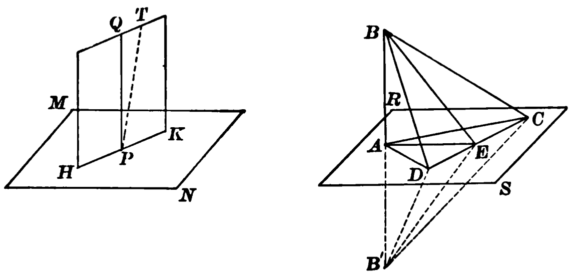
Two planes are said to be *parallel to each other* when they cannot meet however far they may be produced.

398. Sch. The following is given for convenience of reference:

A perpendicular to a plane is perpendicular to every straight line drawn in the plane through its foot.

PROP. III. THEOREM.

399. *At a given point in a plane, one perpendicular to the plane can be drawn, and but one.*



Given point P in plane MN.

To Prove that a ⊥ can be drawn to MN at P, and but one.

Proof. At any point A of indefinite str. line AB, draw lines AC and AD ⊥ to AB.

Let RS be the plane determined by AC and AD.

Let AE be any other str. line drawn through point A in plane RS; and draw line CD intersecting AC, AE, and AD at C, E, and D, respectively.

Produce BA to B', making $AB' = AB$, and draw lines BC, BE, BD, $B'C$, $B'E$, and $B'D$.

In ⊿s BCD and $B'CD$,

$$CD = CD.$$

And since AC and AD are ⊥ to BB' at its middle point,

$$BC = B'C \text{ and } BD = B'D.$$

[If a ⊥ be erected at the middle point of a str. line, any point in the ⊥ is equally distant from the extremities of the line.] (§ 41, I)

$$\therefore \triangle BCD = \triangle B'CD.$$

[Two $\triangle$s are equal when the three sides of one are equal respectively to the three sides of the other.] (§ 69)

Now revolve $\triangle BCD$ about CD as an axis until it coincides with $\triangle B'CD$.

Then, point B will fall at point B', and line BE will coincide with line $B'E$; that is, $BE = B'E$.

Hence, since points A and E are each equally distant from B and B', line AE is $\perp BB'$.

[Two points, each equally distant from the extremities of a str. line, determine a $\perp$ at its middle point.] (§ 43)

But AE is *any* str. line drawn through A in plane RS.

Then, AB is $\perp$ to *every* str. line drawn through A in plane RS.

Whence, AB is $\perp$ to plane RS.

[A str. line is said to be $\perp$ to a plane when it is $\perp$ to every str. line drawn in the plane through its foot.] (§ 397)

Now apply plane RS to plane MN so that point A shall fall at point P; and let AB take the position PQ.

Then, PQ will be $\perp MN$.

Hence, a $\perp$ can be drawn to MN at P

If possible, let PT be another $\perp$ to plane MN at P; and let the plane determined by PQ and PT intersect MN in line HK.

Then, both PQ and PT are $\perp HK$.

[A $\perp$ to a plane is $\perp$ to every str. line drawn in the plane through its foot.] (§ 398)

But this is impossible; for, in plane HKT, only one $\perp$ can be drawn to HK at P.

[At a given point in a str. line, but one $\perp$ to the line can be drawn.] (§ 25)

Then only one $\perp$ can be drawn to MN at P.

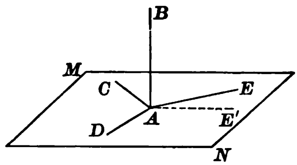
400. Cor. I. *A straight line perpendicular to each of two straight lines at their point of intersection is perpendicular to their plane.*

401. Cor. II. Since E is any point in plane RS, it follows that

If a plane is perpendicular to a straight line at its middle point, any point in the plane is equally distant from the extremities of the line.

Prop. IV. Theorem.

402. *All the perpendiculars to a straight line at a given point lie in a plane perpendicular to the line.*



Given AC, AD, and AE any three ⊥s to line AB at A.

To Prove that they lie in a plane ⊥ to AB.

Proof. Let MN be the plane determined by AC and AD. Then, plane MN is ⊥ AB.

[A str. line ⊥ to each of two str. lines at their point of intersection is ⊥ to their plane.] (§ 400)

Let the plane determined by AB and AE intersect MN in line AE'; then, $AB \perp AE'$.

[A ⊥ to a plane is ⊥ to every str. line drawn in the plane through its foot.] (§ 398)

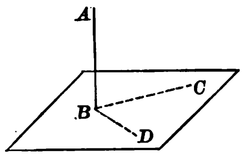
But in plane ABE, only one ⊥ can be drawn to AB at A.

[At a given point in a str. line, but one ⊥ to the line can be drawn.] (§ 25)

Then, AE' coincides with AE, and AE lies in plane MN.
But AC, AD, and AE are any three ⊥s to AB at A.
Therefore, all the ⊥s to AB at A lie in a plane ⊥ AB.

403. Cor. I. *Through a given point in a straight line, a plane can be drawn perpendicular to the line, and but one.*

404. Cor. II. *Through a given point without a straight line, a plane can be drawn perpendicular to the line, and but one.*



Given point C without line AB.

To Prove that a plane can be drawn through $C \perp AB$, and but one.

Proof. Draw line $CB \perp AB$, and let BD be any other $\perp$ to AB at B.

Then, the plane determined by BC and BD will be a plane drawn through $C \perp AB$.

[A str. line $\perp$ to each of two str. lines at their point of intersection is $\perp$ to their plane.] (§ 400)

Again, every plane through $C \perp AB$ must intersect the plane determined by AB and BC in a line from $C \perp AB$.

[A $\perp$ to a plane is $\perp$ to every str. line drawn in the plane through its foot.] (§ 398)

But only one $\perp$ can be drawn from C to AB.

[From a given point without a str. line, but one $\perp$ can be drawn to the line.] (§ 45)

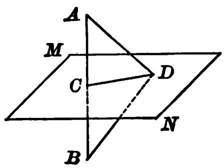
Then, every plane through $C \perp AB$ must contain BC, and be $\perp$ to AB at B.

But only one plane can be drawn through $B \perp AB$.

[Through a given point in a str. line, but one plane can be drawn $\perp$ to the line.] (§ 403)

Hence, but one plane can be drawn through $C \perp AB$.

405. Cor. III. (Converse of § 401.) *Any point equally distant from the extremities of a straight line lies in a plane perpendicular to the line at its middle point.*



Given plane $MN \perp$ to line AB at its middle point C, and point D equally distant from A and B.

To Prove that D lies in MN.

(By § 43, $CD \perp AB$; then use § 402.)

Note. It follows from §§ 401 and 405 that

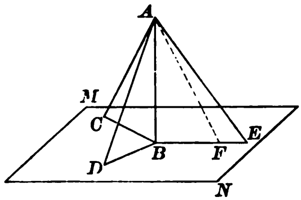
The locus (§ 141) of points in space equally distant from the extremities of a straight line is a plane perpendicular to the line at its middle point.

PROP. V. THEOREM.

406. *If from a point in a perpendicular to a plane, oblique lines be drawn to the plane,*

I. *Two oblique lines cutting off equal distances from the foot of the perpendicular are equal.*

II. *Of two oblique lines cutting off unequal distances from the foot of the perpendicular, the more remote is the greater.*



I. **Given** line $AB \perp$ to plane MN at B, and AC and AD oblique lines meeting MN at equal distances from B.

To Prove $AC = AD$.

Proof. Draw lines BC and BD.

In $\triangle$ ABC and ABD, $AB = AB$.

Also, $\angle ABC = \angle ABD$.

[A $\perp$ to a plane is $\perp$ to every str. line drawn in the plane through its foot.] (§ 398)

And by hyp., $BC = BD$.

$\therefore \triangle ABC = \triangle ABD$.

[Two $\triangle$ are equal when two sides and the included $\angle$ of one are equal respectively to two sides and the included $\angle$ of the other.] (§ 63)

$\therefore AC = AD$.

[In equal figures, the homologous parts are equal.] (§ 66)

II. **Given** line $AB \perp$ to plane MN at B, and AC and AE oblique lines from A to MN, AE meeting MN at a greater distance from B than AC.

To Prove $AE > AC$.

Proof. Draw lines BC and BE.

On BE take $BF = BC$, and draw line AF.

Since AF and AC meet MN at equal distances from B,

$$AF = AC.$$

[If from a point in a $\perp$ to a plane, oblique lines be drawn to the plane, two oblique lines cutting off equal distances from the foot of the $\perp$ are equal.] (§ 406, I)

But, $AB \perp BE$.

[A $\perp$ to a plane is $\perp$ to every str. line drawn in the plane through its foot.] (§ 398)

$$\therefore AE > AF.$$

[If oblique lines be drawn from a point to a str. line, of two oblique lines cutting off unequal distances from the foot of the $\perp$ from the point to the line, the more remote is the greater.] (§ 49, II)

$$\therefore AE > AC.$$

Prop. VI. Theorem.

407. (Converse of Prop. V.) *If from a point in a perpendicular to a plane, oblique lines be drawn to the plane,*

I. *Two equal oblique lines cut off equal distances from the foot of the perpendicular.*

II. *Of two unequal oblique lines, the greater cuts off the greater distance from the foot of the perpendicular.*

I. **Given** line $AB \perp$ to plane MN at B, AC and AD equal oblique lines from A to MN, and lines BC and BD. (Fig. of Prop. V.)

To Prove $BC = BD$.

(Prove ⚠ ABC and ABD equal.)

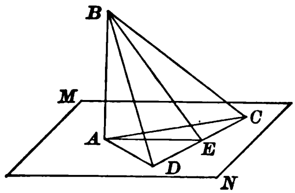
II. **Given** line $AB \perp$ to plane MN at B, and AC and AE oblique lines from A to MN, AE being $> AC$; also, lines BC and BE.

To Prove $BE > BC$.

(The proof is left to the pupil.)

PROP. VII. THEOREM.

408. *If through the foot of a perpendicular to a plane a line be drawn at right angles to any line in the plane, the line drawn from its intersection with this line to any point in the perpendicular will be perpendicular to the line in the plane.*



Given line $AB \perp$ to plane MN at A, line $AE \perp$ to any line CD in MN, and line BE from E to any point B in AB.

To Prove $BE \perp CD$.

Proof. On CD take $EC = ED$.

Draw lines AC, AD, BC, and BD.

$\therefore AC = AD$.

[If a $\perp$ be erected at the middle point of a str. line, any point in the $\perp$ is equally distant from the extremities of the line.] (§ 41, I)

$\therefore BC = BD$.

[If from a point in a $\perp$ to a plane, oblique lines be drawn to the plane, two oblique lines cutting off equal distances from the foot of the $\perp$ are equal.] (§ 406, I)

Then since each of the points B and E is equally distant from C and D,

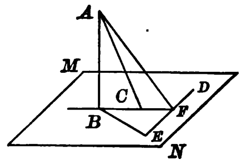
$BE \perp CD$.

[Two points, each equally distant from the extremities of a str. line, determine a $\perp$ at its middle point.] (§ 43)

409. Cor. I. *From a given point without a plane, one perpendicular to the plane can be drawn, and but one.*

Given point *A* without plane *MN*.

To Prove that a ⊥ can be drawn from *A* to *MN*, and but one.



Proof. Let *DE* be any line in plane *MN*; draw line *AF* ⊥ *DE*, line *BF* in plane *MN* ⊥ *DE*, line *AB* ⊥ *BF*, and line *BE*.

Now *EF* is ⊥ to the plane determined by *AF* and *BF*.

[A str. line ⊥ to each of two str. lines at their point of intersection is ⊥ to their plane.] (§ 400)

Then since *BF* is drawn through the foot of *EF*, ⊥ to line *AB* in plane *ABF*, we have *BE* ⊥ *AB*.

[If through the foot of a ⊥ to a plane a line be drawn at rt. ∠s to any line in the plane, the line drawn from its intersection with this line to any point in the ⊥ will be ⊥ to the line in the plane.] (§ 408)

Then *AB*, being ⊥ to *BE* and *BF*, is ⊥ to *MN*.

[A str. line ⊥ to each of two str. lines at their point of intersection is ⊥ to their plane.] (§ 400)

If possible, let *AC* be another ⊥ from *A* to *MN*; then △*ABC* will have two rt. ∠s.

[A ⊥ to a plane is ⊥ to every str. line drawn in the plane through its foot.] (§ 398)

But this is impossible.

Hence, but one ⊥ can be drawn from *A* to *MN*.

410. Cor. II. *The perpendicular is the shortest line that can be drawn from a point to a plane.*

Given *AB* the ⊥ from point *A* to plane *MN*, and *AC* any other str. line from *A* to *MN*. (Fig. of § 409.)

To Prove $AB < AC$.

Proof. Draw line *BC*; then, *AB* ⊥ *BC*.

[A ⊥ to a plane is ⊥ to every str. line drawn in the plane through its foot.] (§ 398)

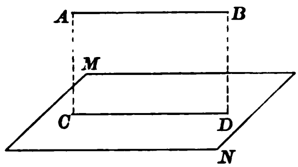
$$\therefore AB < AC.$$

[The ⊥ is the shortest line that can be drawn from a point to a str. line.] (§ 46)

Note. The *distance* of a point from a plane signifies the length of the perpendicular from the point to the plane.

PROP. VIII. THEOREM.

411. *If two straight lines are parallel, a plane drawn through one of them, not coinciding with the plane of the parallels, is parallel to the other.*



Given line $AB \parallel$ to line CD, and plane MN drawn through CD, not coinciding with the plane of the $\parallel$s.

To Prove $AB \parallel MN$.

Proof. The $\parallel$s AB and CD lie in a plane which intersects MN in line CD.

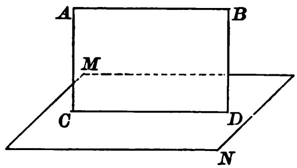
Hence, if AB meets MN, it must be at some point of CD.

But since AB is $\parallel CD$, it cannot meet CD (§ 52).

Then AB and MN cannot meet, and are $\parallel$ (§ 397).

PROP. IX. THEOREM.

412. *If a straight line is parallel to a plane, the intersection of the plane with any plane drawn through the line is parallel to the line.*



Given line $AB \parallel$ to plane MN; and line CD the intersection of MN with any plane AD drawn through AB.

To Prove $AB \parallel CD$.

(AB and CD lie in the same plane, and cannot meet.)

413. Cor. *If a line and a plane are parallel, a parallel to the line through any point of the plane lies in the plane.*

Given line $AB \parallel$ to plane MN; and line CD through any point C of $MN \parallel$ to AB. (Fig. of Prop. IX.)

To Prove that CD lies in MN.

Proof. The plane determined by line AB and point C intersects MN in a line $\parallel$ to AB.

[If a str. line is $\parallel$ to a plane, the intersection of the plane with any plane drawn through the line is $\parallel$ to the line.] (§ 412)

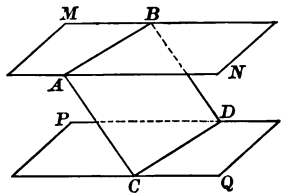
But through C, only one $\parallel$ can be drawn to AB.

[But one str. line can be drawn through a given point $\parallel$ to a given str. line.] (§ 53)

Whence, CD lies in MN.

PROP. X. THEOREM.

414. *If two parallel planes are cut by a third plane, the intersections are parallel.*



Given $\parallel$ planes MN and PQ cut by plane AD in lines AB and CD, respectively.

To Prove $AB \parallel CD$.

(AB and CD lie in the same plane, and cannot meet.)

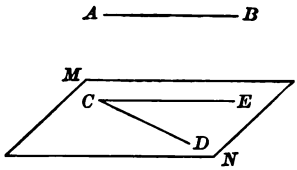
415. Cor. *Parallel lines included between parallel planes are equal.*

Given AC and $BD \parallel$ lines included between $\parallel$ planes MN and PQ. (Fig. of Prop. X.)

(Prove $AC = BD$ by §§ 414 and 107.)

PROP. XI. THEOREM.

416. *Through any given straight line, a plane can be drawn parallel to any other straight line.*



Given lines AB and CD.

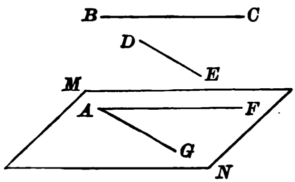
To Prove that a plane can be drawn through $CD \parallel AB$.

(Draw line $CE \parallel AB$; then use § 411.)

Note. If AB is $\parallel$ CD, an indefinitely great number of planes can be drawn through $CD \parallel AB$ (§ 411); otherwise, but one such plane can be drawn, for every plane drawn through $CD \parallel AB$ must contain CE (§ 413), and but one plane can be drawn through CD and CE.

PROP. XII. THEOREM.

417. *Through a given point a plane can be drawn parallel to any two straight lines in space.*



Given point A and lines BC and DE.

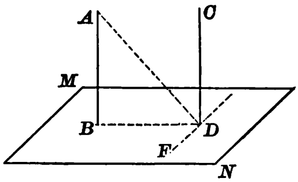
To Prove that a plane can be drawn through A $\parallel$ to BC and DE.

(The proof is left to the pupil; see § 411.)

Note. If BC and DE are $\parallel$, an indefinitely great number of planes can be drawn through A $\parallel$ to BC and DE (§ 411); otherwise, but one such plane can be drawn.

PROP. XIII. THEOREM.

418. *Two perpendiculars to the same plane are parallel.*



Given lines *AB* and *CD* ⊥ to plane *MN* at *B* and *D*, respectively.

To Prove *AB* ∥ *CD*.

Proof. Let *A* be any point of *AB*, and draw line *AD*.

Also, draw line *BD*, and line *DF* in plane *MN* ⊥ *BD*.

∴ *CD* ⊥ *DF*.

[A ⊥ to a plane is ⊥ to every str. line drawn in the plane through its foot.] (§ 398)

Also, *AD* ⊥ *DF*.

[If through the foot of a ⊥ to a plane a line be drawn at rt. ∠s to any line in the plane, the line drawn from its intersection with this line to any point in the ⊥ will be ⊥ to the line in the plane.] (§ 408)

Then, *CD*, *AD*, and *BD*, being ⊥ to *DF* at *D*, lie in the same plane.

[All the ⊥s to a str. line at a given point lie in a plane ⊥ to the line.] (§ 402)

Then, since points *A* and *B* lie in the plane of the lines *AD*, *BD*, and *CD*, *AB* lies in this plane.

[A plane is a surface such that the str. line joining any two of its points lies entirely in the surface.] (§ 9)

That is, *AB* and *CD* lie in the same plane.

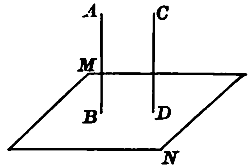
Again, *AB* and *CD* are ⊥ *BD*.

[A ⊥ to a plane is ⊥ to every str. line drawn in the plane through its foot.] (§ 398)

∴ *AB* ∥ *CD*.

[Two ⊥s to the same str. line are ∥.] (§ 54)

419. Cor. I. *If one of two parallel lines is perpendicular to a plane, the other is also perpendicular to the plane.*



Given lines AB and CD ∥, and AB ⊥ to plane MN.

To Prove $CD \perp MN$.

Proof. A ⊥ from C to MN will be ∥ AB.

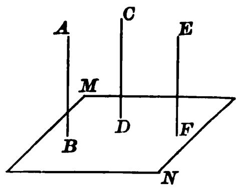
[Two ⊥s to the same plane are ∥.] (§ 418)

But through C, only one ∥ can be drawn to AB.

[But one str. line can be drawn through a given point ∥ to a given str. line.] (§ 53)

$\therefore CD \perp MN$.

420. Cor. II. *If each of two straight lines is parallel to a third straight line, they are parallel to each other.*



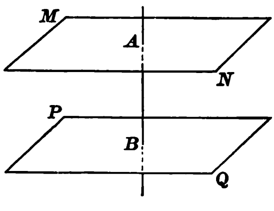
Given lines AB and CD ∥ line EF.

To Prove $AB \parallel CD$.

(Draw plane $MN \perp EF$, and prove $AB \parallel CD$ by §§ 418 and 419.)

PROP. XIV. THEOREM.

421. *Two planes perpendicular to the same straight line are parallel.*



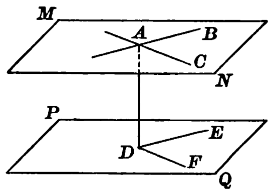
Given planes MN and PQ ⊥ to line AB.

To Prove $MN \parallel PQ$.

(Prove as in § 54; by § 404, but one plane can be drawn through a given point ⊥ to a given str. line.)

Prop. XV. Theorem.

422. *If each of two intersecting lines is parallel to a plane, their plane is parallel to the given plane.*



Given lines AB and AC, in plane MN, ∥ to plane PQ.

To Prove $MN \parallel PQ$.

Proof. Draw line $AD \perp PQ$, and lines DE and DF ∥ to AB and AC, respectively; then DE and DF lie in plane PQ.

[If a line and a plane are ∥, a ∥ to the line through any point of the plane lies in the plane.] (§ 413)

Whence, AD is ⊥ to DE and DF.

[A ⊥ to a plane is ⊥ to every str. line drawn in the plane through its foot.] (§ 398)

Therefore, AD is ⊥ to AB and AC.

[A str. line ⊥ to one of two ∥s is ⊥ to the other.] (§ 56)

$$\therefore AD \perp MN.$$

[A str. line ⊥ to each of two str. lines at their point of intersection is ⊥ to their plane.] (§ 400)

$$\therefore MN \parallel PQ.$$

[Two planes ⊥ to the same str. line are ∥.] (§ 421)

EXERCISES.

1. What is the locus (§ 141) of the perpendiculars to a given straight line at a given point?

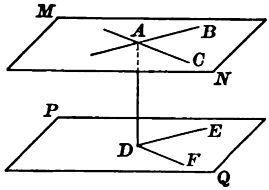
2. What is the locus of points in space equally distant from the circumference of a given circle?

3. A line parallel to a plane is everywhere equally distant from it.

(Fig. of Prop. IX. Draw lines AC and $BD \perp MN$. To prove $AC = BD$.)

PROP. XVI. THEOREM.

423. *A straight line perpendicular to one of two parallel planes is perpendicular to the other also.*



Given MN and PQ ∥ planes, and line $AD \perp PQ$.

To Prove $AD \perp MN$.

Proof. Pass two planes through AD, intersecting MN in lines AB and AC, and PQ in lines DE and DF, respectively.

Then, $AB \parallel DE$, and $AC \parallel DF$.

[If two ∥ planes are cut by a third plane, the intersections are ∥.] (§ 414)

But AD is ⊥ to DE and DF.

[A ⊥ to a plane is ⊥ to every str. line drawn in the plane through its foot.] (§ 398)

Whence, AD is ⊥ to AB and AC.

[A str. line ⊥ to one of two ∥s is ⊥ to the other.] (§ 56)

$\therefore AD \perp MN$.

[A str. line ⊥ to each of two str. lines at their point of intersection is ⊥ to their plane.] (§ 400)

424. Cor. I. *Two parallel planes are everywhere equally distant.* (Note, p. 244.)

Given MN and PQ ∥ planes. (Fig. of Prop. XVI.)

To Prove MN and PQ everywhere equally distant.

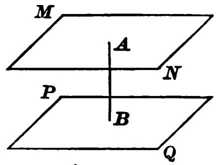
Proof. All lines which are ⊥ to both planes are ∥.

[Two ⊥s to the same plane are ∥.] (§ 418)

Therefore, these lines are all equal.

[∥ lines included between ∥ planes are equal.] (§ 415)

425. Cor. II. *Through a given point a plane can be drawn parallel to a given plane, and but one.*



Given point A and plane PQ.

To Prove that a plane can be drawn through $A \parallel PQ$, and but one.

Proof. Draw line $AB \perp PQ$.

Through A pass plane $MN \perp AB$.

Then MN will be $\parallel PQ$.

[Two planes $\perp$ to the same str. line are $\parallel$.] (§ 421)

If another plane could be drawn through $A \parallel PQ$, it would be $\perp AB$.

[A str. line $\perp$ to one of two $\parallel$ planes is $\perp$ to the other also.] (§ 423)

It would then coincide with MN.

[Through a given point in a str. line, but one plane can be drawn $\perp$ to the line.] (§ 403)

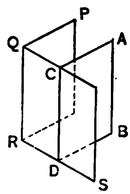
Then but one plane can be drawn through $A \parallel PQ$.

EXERCISES.

4. What is the locus of points in space equally distant from the vertices of a given triangle?

5. What is the locus of points in space equally distant from a given plane?

6. What is the locus of points in space equally distant from two parallel planes?



7. A line parallel to each of two intersecting planes is parallel to their intersection.

(Pass a plane through $AB \parallel PR$; then use § 412.)

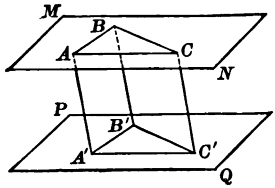
8. If two planes are parallel to a third plane, they are parallel to each other. (§§ 423, 421.)

9. Line AB is perpendicular to plane MN at A. A line is drawn from A meeting any line CD of plane MN at E. If line BE is perpendicular to CD, prove AE perpendicular to CD.

(Fig. of Prop. VII.)

PROP. XVII. THEOREM.

426. *If two angles not in the same plane have their sides parallel and extending in the same direction, they are equal, and their planes are parallel.*



Given ∠s BAC and $B'A'C'$ in planes MN and PQ, respectively, with AB and AC ∥ respectively to $A'B'$ and $A'C'$, and extending in the same direction.

To Prove $\angle BAC = \angle B'A'C'$, and $MN \parallel PQ$.

Proof. Lay off $AB = A'B'$ and $AC = A'C'$, and draw lines AA', BB', CC', BC, and $B'C'$.

Then since AB is equal and ∥ to $A'B'$, $ABB'A'$ is a ▱.

[If two sides of a quadrilateral are equal and ∥, the figure is a ▱.] (§ 110)

Whence, AA' is equal and ∥ to BB'.

[The opposite sides of a ▱ are equal.] (§ 106, I)

Similarly, $ACC'A'$ is a ▱, and AA' is equal and ∥ to CC'.

Then, BB' is equal and ∥ to CC'.

[If each of two str. lines is ∥ to a third str. line, they are ∥ to each other.] (§ 420)

Whence, $BB'C'C$ is a ▱, and $BC = B'C'$.

$$\therefore \triangle ABC = \triangle A'B'C'.$$

[Two △s are equal when the three sides of one are equal respectively to the three sides of the other.] (§ 69)

$$\therefore \angle BAC = \angle B'A'C'.$$

[In equal figures, the homologous parts are equal.] (§ 66)

Again, lines AB and AC are ∥ to plane PQ.

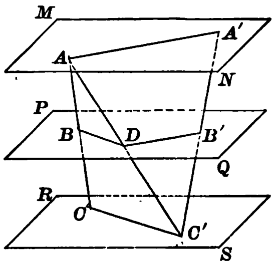
[If two str. lines are ∥, a plane drawn through one of them, not coinciding with the plane of the ∥s, is ∥ to the other.] (§ 411)

$\therefore MN \parallel PQ.$

[If each of two intersecting lines is ∥ to a plane, their plane is ∥ to the given plane.] (§ 422)

Prop. XVIII. Theorem.

427. *If two straight lines are cut by three parallel planes, the corresponding segments are proportional.*



Given ∥ planes MN, PQ, and RS intersecting lines AC and $A'C'$ in points A, B, C, and A', B', C', respectively.

To Prove $$\frac{AB}{BC}=\frac{A'B'}{B'C'}.$$

Proof. Draw line AC'; and through AC and AC' pass a plane intersecting PQ and RS in lines BD and CC', respectively.

$$\therefore BD \parallel CC'.$$

[If two ∥ planes are cut by a third plane, the intersections are ∥.] (§ 414)

$$\therefore \frac{AB}{BC}=\frac{AD}{DC'}. \quad (1)$$

[A ∥ to one side of a △ divides the other two sides proportionally.] (§ 244)

In like manner, $$\frac{AD}{DC'}=\frac{A'B'}{B'C'}. \quad (2)$$

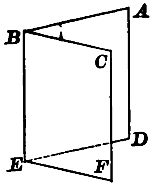
From (1) and (2), $$\frac{AB}{BC}=\frac{A'B'}{B'C'}.$$

[Things which are equal to the same thing, are equal to each other.] (Ax. 1)

DIEDRAL ANGLES.

DEFINITIONS.

428. A *diedral angle* is the *amount of divergence* of two planes which meet in a straight line.

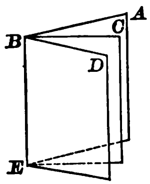


The line of intersection of the planes is called the *edge* of the diedral angle, and the planes are called its *faces.*

Thus, in the diedral angle between planes *BD* and *BF*, *BE* is the edge, and *BD* and *BF* the faces.

A diedral angle may be designated by two letters on its edge; or, if several diedral angles have a common edge, by four letters, one in each face and two on the edge, the letters on the edge being named between the other two.

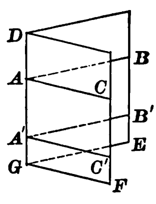
Thus, we may read the above diedral angle *BE*, or *ABEC*.



Two diedral angles are said to be *adjacent* when they have the same edge, and a common face between them; as, *ABEC* and *CBED*.

Two diedral angles are said to be *vertical* when the faces of one are the extensions of the faces of the other.

429. A *plane angle* of a diedral angle is the angle between two straight lines drawn one in each face, perpendicular to the edge at the same point.



Thus, if lines *AB* and *AC* be drawn in faces *DE* and *DF*, respectively, of diedral angle *DG*, perpendicular to *DG* at *A*, $\angle BAC$ is a plane angle of the diedral angle.

430. Let *BAC* and *B'A'C'* (Fig. of § 429) be plane $\measuredangle$ of diedral $\angle DG$; then, $AB \parallel A'B'$ and $AC \parallel A'C'$. (§ 54)

$$\therefore \angle BAC = \angle B'A'C'. \quad (\S\ 426)$$

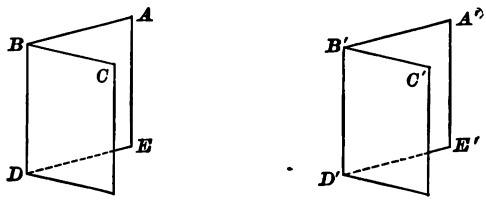
That is, *all plane angles of a diedral angle are equal.*

431. A plane perpendicular to the edge of a diedral angle intersects the faces in lines perpendicular to the edge (§ 398); hence, *a plane perpendicular to the edge of a diedral angle intersects the faces in lines which include the plane angle of the diedral angle* (§ 429).

432. Two diedral angles are *equal* when their faces may be made to coincide.

PROP. XIX. THEOREM.

433. *Two diedral angles are equal if their plane angles are equal.*



Given ABC and $A'B'C'$ plane $\angle$s of diedral $\angle$s BD and $B'D'$, respectively, and $\angle ABC = \angle A'B'C'$.

To Prove diedral $\angle BD =$ diedral $\angle B'D'$.

Proof. Apply diedral $\angle B'D'$ to BD in such a way that $A'B'$ shall coincide with AB, and $B'C'$ with BC.

Now BD and $B'D'$ are $\perp$ to the planes of $\angle$s ABC and $A'B'C'$, respectively. (§ 400)

Whence, $B'D'$ will coincide with BD. (§ 399)

Then, $A'D'$ will coincide with AD, and $C'D'$ with CD. (§ 395, III)

Hence, $B'D'$ and BD are equal. (§ 432)

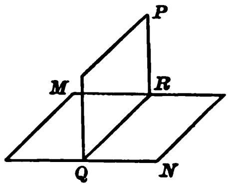
434. Cor. I. (Converse of Prop. XIX.) *If two diedral angles are equal, their plane angles are equal.* (Fig. of Prop. XIX.)

(Apply $B'D'$ to BD so that face $A'D'$ shall coincide with AD, and $C'D'$ with CD, point B' falling at B.)

435. Cor. II. *If two planes intersect, the vertical diedral angles are equal.*

For their plane angles are equal. (§ 40)

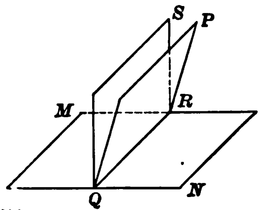
436. Defs. If a plane meets another plane in such a way as to make the adjacent diedral angles equal, each is called a *right diedral angle*, and the planes are said to be *perpendicular to each other.*



Thus, if plane PQ be drawn meeting plane MN in such a way as to make diedral $\measuredangle s$ $PRQM$ and $PRQN$ equal, each of these is a right diedral $\angle$, and MN and PQ are $\perp$ to each other.

PROP. XX. THEOREM.

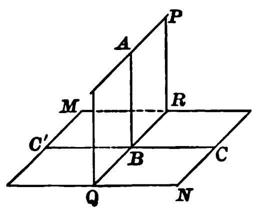
437. *Through a given line in a plane, a plane can be drawn perpendicular to the given plane, and but one.*



(Prove as in § 25.)

PROP. XXI. THEOREM.

438. *If two planes are perpendicular to each other, a straight line drawn in one of them perpendicular to their intersection is perpendicular to the other.*



Given planes PQ and $MN \perp$, intersecting in line QR, and line AB in plane $PQ \perp QR$.

To Prove $AB \perp MN$.

Proof. Draw line $C'BC$ in plane $MN \perp QR$.

Then, ABC and ABC' are plane ∠s of diedral ∠s $PRQN$ and $PRQM$, respectively. (§ 429)

Now, if two planes are ⊥ to each other, the adj. diedral ∠s are equal (§ 436).

That is, diedral $\angle PRQN =$ diedral $\angle PRQM$.

$$\therefore \angle ABC = \angle ABC'. \qquad (§\ 434)$$

Whence, $\angle ABC$ is a rt. ∠. (§ 24)

Then AB, being ⊥ to BC and BQ at B, is $\perp MN$. (§ 400)

439. Cor. I. *If two planes are perpendicular to each other, a perpendicular to one of them at any point of their intersection lies in the other.*

Given planes PQ and $MN \perp$, intersecting in line QR, and line AB drawn from any point B of $QR \perp MN$. (Fig. of Prop. XXI.)

To Prove that AB lies in PQ.

Proof. If a line be drawn in PQ from point $B \perp QR$, it will be $\perp MN$. (§ 438)

But from point B but one ⊥ can be drawn to MN. (§ 399)

Therefore, AB lies in PQ.

440. Cor. II. *If two planes are perpendicular to each other, a perpendicular to one of them from any point of the other lies in the other.*

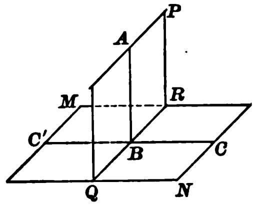
Given planes PQ and $MN \perp$, intersecting in line QR, and line AB drawn from any point A of $PQ \perp MN$. (Fig. of Prop. XXI.)

To Prove that AB lies in PQ.

(The proof is left to the pupil.)

PROP. XXII. THEOREM.

441. *If a straight line is perpendicular to a plane, every plane drawn through the line is perpendicular to the plane.*



Given line $AB \perp$ plane MN, and PQ any plane drawn through AB.

To Prove $PQ \perp MN$.

Proof. Let line QR be the intersection of PQ and MN, and draw line $C'BC$ in plane $MN \perp QR$.

We have $AB \perp BQ$. (§ 398)

Then, $\angle$s ABC and ABC' are plane $\angle$s of diedral $\angle$s $PRQN$ and $PRQM$, respectively. (§ 429)

But $\angle$s ABC and ABC' are rt. $\angle$s. (§ 398)

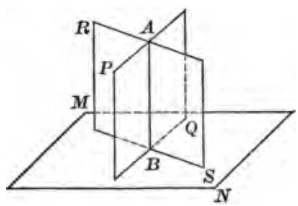
$\therefore \angle ABC = \angle ABC'$. (§ 26)

$\therefore$ diedral $\angle PRQN =$ diedral $\angle PRQM$. (§ 433)

$\therefore PQ \perp MN$. (§ 436)

PROP. XXIII. THEOREM.

442. *A plane perpendicular to each of two intersecting planes is perpendicular to their intersection.*



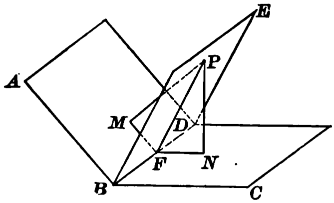
Given planes PQ and $RS \perp$ to plane MN, and intersecting in line AB.

To Prove $AB \perp MN$.

(By § 439, a $\perp$ to MN at B lies in both PQ and RS.)

PROP. XXIV. THEOREM.

443. *Every point in the bisecting plane of a diedral angle is equally distant from its faces.*



Given P any point in bisecting plane BE of diedral $\angle ABDC$, and lines PM and $PN \perp$ to AD and CD, respectively.

To Prove $PM = PN$.

Proof. Let the plane determined by PM and PN intersect planes AD, BE, and CD in lines FM, FP, and FN, respectively.

Plane $PMFN$ is $\perp$ to planes AD and CD. (§ 441)

Then, plane $PMFN$ is $\perp BD$. (§ 442)

Whence, PFM and PFN are plane $\angle$s of diedral $\angle$s $ABDE$ and $CBDE$, respectively. (§ 431)

$\therefore \angle PFM = \angle PFN$. (§ 434)

In $\triangle$s PFM and PFN, $PF = PF$.

And, $\angle PFM = \angle PFN$.

Also, $\angle$s PMF and PNF are rt. $\angle$s. (§ 398)

$\therefore \triangle PFM = \triangle PFN$. (§ 70)

$\therefore PM = PN$. (?)

444. Cor. I. (Converse of Prop. XXIV.) *Any point which is within a diedral angle, and equally distant from its faces, lies in the bisecting plane of the diedral angle.*

Given point P within diedral $\angle ABDC$, equally distant from AD and CD, and plane BE determined by BD and P. (Fig. of Prop. XXIV.)

To Prove that BE bisects diedral $\angle ABDC$.

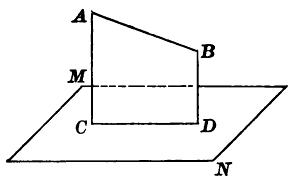
(Prove $\triangle$ PFM and PFN equal; then $\angle PFM = \angle PFN$, and the theorem follows by § 433.)

445. Cor. II. It follows from §§ 443 and 444 that

The locus of points in space equally distant from the faces of a diedral angle is the plane bisecting the diedral angle.

Prop. XXV. Theorem.

446. *Through a given straight line without a plane, a plane can be drawn perpendicular to the given plane, and but one.*



Given line AB without plane MN.

To Prove that a plane can be drawn through $AB \perp MN$, and but one.

Proof. Draw line $AC \perp MN$, and let AD be the plane determined by AB and AC; then, $AD \perp MN$. (§ 441)

If more than one plane could be drawn through $AB \perp MN$, their common intersection, AB, would be $\perp MN$. (§ 442)

Hence, but one plane can be drawn through $AB \perp MN$, unless AB is $\perp MN$.

Note. If line AB is $\perp MN$, an indefinitely great number of planes can be drawn through $AB \perp MN$ (§ 441).

447. Defs. *The projection of a point on a plane* is the foot of the perpendicular drawn from the point to the plane.

The *projection of a line on a plane* is the line which contains the projections of all its points.

448. Cor. *The projection of a straight line on a plane is a straight line.*

Given line CD the projection (§ 447) of str. line AB on plane MN. (Fig. of Prop. XXV.)

To Prove CD a str. line.

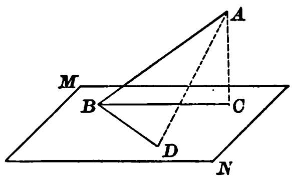
Proof. Draw a plane through $AB \perp MN$.

The $\perp$s to MN from all points of AB will lie in this plane. (§ 440)

Therefore, CD is a str. line. (§ 396)

PROP. XXVI. THEOREM.

449. *The angle between a straight line and its projection on a plane is the least angle which it makes with any line drawn in the plane through its foot.*



Given line BC the projection of line AB on plane MN, and BD any other line drawn through B in MN.

To Prove $\angle ABC < \angle ABD$.

Proof. Lay off $BD = BC$, and draw lines AC and AD.

In $\triangle$s ABC and ABD, $AB = AB$.

And by hyp., $BC = BD$.

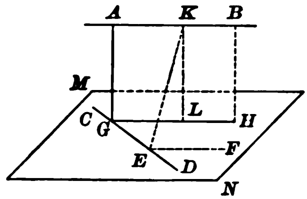
Also, $AC < AD$. (§ 410)

$\therefore \angle ABC < \angle ABD$. (§ 92)

Note. $\angle ABC$ is called the *angle* between line AB and plane MN.

PROP. XXVII. THEOREM.

450. *Two straight lines, not in the same plane, have one common perpendicular, and but one; and this line is the shortest line that can be drawn between them.*



Given lines AB and CD, not in the same plane.

To Prove that one common ⊥ to AB and CD can be drawn, and but one; and that this line is the shortest line that can be drawn between AB and CD.

Proof. Through CD draw plane $MN \parallel AB$. (§ 416)

Through AB draw plane $AH \perp MN$, and produce their intersection to meet CD at G. (§ 446)

Draw line AG in plane $AH \perp GH$; then, $AG \perp MN$. (§ 438)

$\therefore AG \perp CD$. (§ 398)

Also, $GH \parallel AB$. (§ 412)

$\therefore AG \perp AB$. (§ 56)

Then, AG is a common ⊥ to AB and CD.

If possible, let EK be another common ⊥ to AB and CD, and draw line $EF \parallel AB$, and line KL in plane $AH \perp GH$.

Then, EF lies in plane MN. (§ 413)

Also, EK is ⊥ to ED and EF. (§ 56)

Whence, EK is ⊥ MN. (?)

But KL is also ⊥ MN. (§ 438)

We should then have two ⊥s from K to MN, which is impossible. (§ 409)

Hence, but one common ⊥ can be drawn to AB and CD.

Again, $EK > KL$. (§ 410)

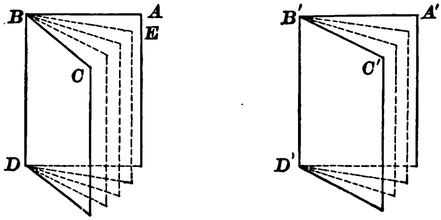
$$\therefore\ EK > AG. \qquad (\S\ 80)$$

Hence, AG is the shortest line between AB and CD.

Prop. XXVIII. Theorem.

451. *Two diedral angles are to each other as their plane angles.*

Case I. *When the plane angles are commensurable.*



Given ABC and $A'B'C'$, plane $\angle$s of diedral $\angle$s $ABDC$ and $A'B'D'C'$, respectively, and commensurable.

To Prove $$\frac{ABDC}{A'B'D'C'} = \frac{\angle ABC}{\angle A'B'C'}.$$

Proof. Let $\angle ABE$ be a common measure of $\angle$s ABC and $A'B'C'$; and suppose it to be contained 4 times in $\angle ABC$ and 3 times in $\angle A'B'C'$.

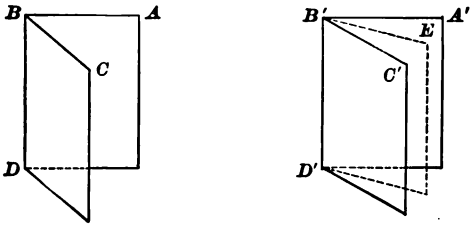
$$\therefore\ \frac{\angle ABC}{\angle A'B'C'} = \frac{4}{3}. \qquad (1)$$

Passing planes through edges BD and $B'D'$, and the several lines of division of $\angle$s ABC and $A'B'C'$, respectively, diedral $\angle ABDC$ will be divided into 4 parts, and diedral $\angle A'B'D'C'$ into 3 parts, all of which parts are equal. (§ 433)

$$\therefore\ \frac{ABDC}{A'B'D'C'} = \frac{4}{3}. \qquad (2)$$

From (1) and (2), $$\frac{ABDC}{A'B'D'C'} = \frac{\angle ABC}{\angle A'B'C'}. \qquad (?)$$

Case II. *When the plane angles are incommensurable.*



Given ABC and $A'B'C'$ plane $\angle$s of diedral $\angle$s $ABDC$ and $A'B'D'C'$, respectively, and incommensurable.

To Prove $$\frac{ABDC}{A'B'D'C'} = \frac{\angle ABC}{\angle A'B'C'}.$$

Proof. Let $\angle ABC$ be divided into any number of equal parts, and let one of these parts be applied to $\angle A'B'C'$ as a unit of measure.

Since $\angle$s ABC and $A'B'C'$ are incommensurable, a certain number of the parts will extend from $A'B'$ to $B'E$, leaving a remainder $\angle EB'C' <$ one of the equal parts.

Pass a plane through $B'D'$ and $B'E$; then since the plane $\angle$s of diedral $\angle$s $A'B'D'E$ and $ABDC$ are commensurable,

$$\frac{ABDC}{A'B'D'E} = \frac{\angle ABC}{\angle A'B'E}. \qquad \text{(§ 451, Case I)}$$

Now let the number of subdivisions of $\angle ABC$ be indefinitely increased.

Then the unit of measure will be indefinitely diminished, and the remainder $\angle EB'C'$ will approach the limit 0.

Then $\dfrac{ABDC}{A'B'D'E}$ will approach the limit $\dfrac{ABDC}{A'B'D'C'}$,

and $\dfrac{\angle ABC}{\angle A'B'E}$ will approach the limit $\dfrac{\angle ABC}{\angle A'B'C'}$.

By the Theorem of Limits, these limits are equal. (§ 188)

$$\therefore \frac{ABDC}{A'B'D'C'} = \frac{\angle ABC}{\angle A'B'C'}.$$

Note. It follows from § 451 that the plane angle may be taken as the *measure* of the diedral angle; thus, if the plane angle contains n degrees, the diedral angle may be regarded as being of n degrees.

EXERCISES.

10. A straight line and a plane perpendicular to the same straight line are parallel.

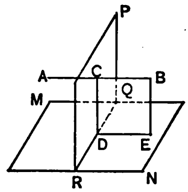
(Fig. of Prop. IX. Let plane determined by AB and AC intersect MN in CD.)

11. If two planes are parallel, a line parallel to one of them through any point of the other lies in the other.

(Fig. of Prop. X. Given planes MN and PQ ||, and AB through any point A of $MN \parallel PQ$. Prove that AB lies in MN by § 413.)

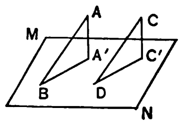
12. If a straight line is parallel to a plane, any plane perpendicular to the line is perpendicular to the plane.

(Draw line $CD \perp QR$, and prove it $\perp MN$.)



13. If two parallels meet a plane, they make equal angles with it.

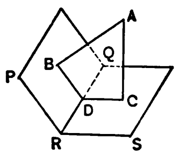
(Given $AB \parallel CD$; to prove $\angle ABA' = \angle CDC'$.)



14. If a straight line intersects two parallel planes, it makes equal angles with them.

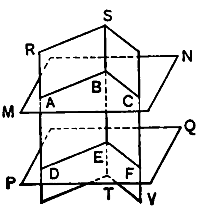
15. The angle between perpendiculars to the faces of a diedral angle from any point within the angle is the supplement of its plane angle.

(Prove $\angle BDC$ the plane $\angle$ of diedral $\angle PQRS$.)



16. If each of two intersecting planes be cut by two parallel planes, not parallel to their intersection, their intersections with the parallel planes include equal angles.

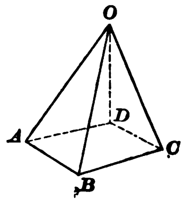
(To prove $\angle ABC = \angle DEF$.)



POLYEDRAL ANGLES.

DEFINITIONS.

452. A *polyedral angle* is a figure composed of three or more triangles, called *faces*, having for their bases the sides of a polygon, and for their common vertex a point without its plane; as *O–ABCD*.



The common vertex, *O*, is called the *vertex* of the polyedral angle, and the polygon, *ABCD*, the *base;* the vertical angles of the triangles, *AOB*, *BOC*, etc., are called the *face angles*, and their sides, *OA*, *OB*, etc., the *edges*.

Note. The polyedral angle is not regarded as limited by the base; thus, the face *AOB* is understood to mean the indefinite plane between the edges *OA* and *OB* produced indefinitely.

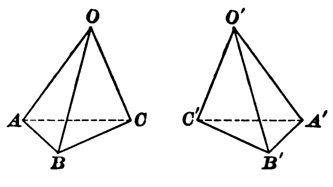
A *triedral angle* is a polyedral angle of three faces.

Two polyedral angles are called *vertical* when the edges of one are the prolongations of the edges of the other.

453. A polyedral angle is called *convex* when its base is a convex polygon (§ 121).

454. Two polyedral angles are *equal* when they can be applied to each other so that their faces shall coincide.

455. Two polyedral angles are said to be *symmetrical* when the face and diedral angles of one are equal respectively to the homologous face and diedral angles of the other, if the equal parts occur *in the reverse order*.



Thus, if face $\angle$s *AOB*, *BOC*, and *COA* are equal respectively to face $\angle$s *A′O′B′*, *B′O′C′*, and *C′O′A′*, and diedral $\angle$s *OA*, *OB*, and *OC* to diedral $\angle$s *O′A′*, *O′B′*, and *O′C′*, triedral $\angle$s *O–ABC* and *O′–A′B′C′* are symmetrical.

It is evident that, in general, two symmetrical polyedral angles cannot be placed so that their faces shall coincide.

Prop. XXIX. Theorem.

456. *Two vertical polyedral angles are symmetrical.*

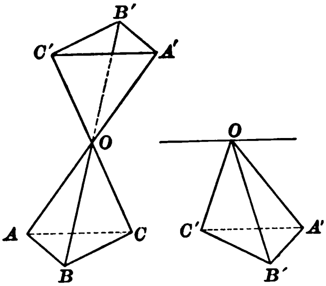


Fig. 1. *Fig. 2.*

Given O–ABC and O–$A'B'C'$ (Fig. 1) vertical triedral $\angle$s.

To Prove O–ABC and O–$A'B'C'$ symmetrical.

Proof. Face $\angle$s AOB, BOC, etc., are equal, respectively, to face $\angle$s $A'OB'$, $B'OC'$, etc. (§ 40)

Again, diedral $\angle$s OA and OA' are vertical; for AOB and $A'OB'$ are portions of the same plane, as also are AOC and $A'OC'$; in like manner, diedral $\angle$s OB and OB' are vertical; etc.

Then, diedral $\angle$s OA, OB, etc., are equal, respectively, to diedral $\angle$s OA', OB', etc. (§ 435)

But the equal parts of the triedral $\angle$s occur *in the reverse order;* as may be seen by conceiving O–$A'B'C'$ moved ∥ to itself to the right, and then revolved, as shown in Fig. 2, about an axis passing through O, until face $OA'C'$ comes into the same plane as before; edge OB' being on this side of, instead of beyond, plane $OA'C'$.

Hence, O–ABC and O–$A'B'C'$ are symmetrical (§ 455).

In like manner, the theorem may be proved for any two polyedral $\angle$s.

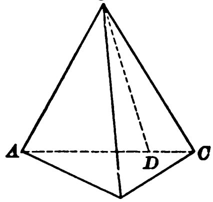
Ex. 17. If two parallel planes are cut by a third plane, the alternate-interior diedral angles are equal.

(Prove the plane ∠s of the alt.-int. diedral ∠s equal.)

Prop. XXX. Theorem.

457. *The sum of any two face angles of a triedral angle is greater than the third.*

Note. The theorem requires proof only in the case where the third face angle is greater than either of the others.



Given in triedral $\angle O\text{-}ABC$,

face $\angle AOC >$ face $\angle AOB$ or face $\angle BOC$.

To Prove $\angle AOB + \angle BOC > \angle AOC$.

Proof. In face AOC draw line OD equal to OB, making $\angle AOD = \angle AOB$; and through B and D pass a plane cutting the faces of the triedral $\angle$ in lines AB, BC, and CA, respectively.

In $\triangle$s AOB and AOD, $OA = OA$.

And by cons., $OB = OD$,

and $\angle AOB = \angle AOD$.

$\therefore \triangle AOB = \triangle AOD$. (?)

$\therefore AB = AD$. (?)

Now, $AB + BC > AD + DC$. (Ax. 4)

Or, since $AB = AD$, $BC > DC$.

Then, in $\triangle$s BOC and COD, $OC = OC$.

Also, $OB = OD$, and $BC > DC$.

$\therefore \angle BOC > \angle COD$. (§ 91)

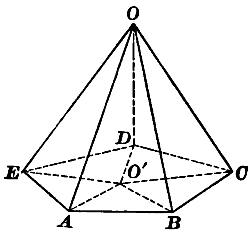
Adding $\angle AOB$ to the first member of this inequality, and its equal $\angle AOD$ to the second member, we have

$$\angle AOB + \angle BOC > \angle AOD + \angle COD.$$

$$\therefore \angle AOB + \angle BOC > \angle AOC.$$

Prop. XXXI. Theorem.

458. *The sum of the face angles of any convex polyedral angle is less than four right angles.*



Given O–$ABCDE$ a convex polyedral $\angle$.

To Prove $\angle AOB + \angle BOC +$ etc. < 4 rt. $\angle$s.

Proof. Let $ABCDE$ be the base of the polyedral $\angle$.

Let O' be any point within polygon $ABCDE$, and draw lines $O'A$, $O'B$, $O'C$, $O'D$, and $O'E$.

Then, in triedral $\angle A$–EOB,

$$\angle OAE + \angle OAB > \angle O'AE + \angle O'AB. \quad (§\ 457)$$

Also, $\angle OBA + \angle OBC > \angle O'BA + \angle O'BC$; etc.

Adding these inequalities, we have the sum of the base $\angle$s of the $\triangle$s whose common vertex is $O >$ the sum of the base $\angle$s of the $\triangle$s whose common vertex is O'.

But the sum of *all* the $\angle$s of the $\triangle$s whose common vertex is O is equal to the sum of *all* the $\angle$s of the $\triangle$s whose common vertex is O'. (§ 84)

Hence, the sum of the $\angle$s at O is $<$ the sum of the $\angle$s at O'.

Then, the sum of the $\angle$s at O is < 4 rt. $\angle$s. (§ 35)

PROP. XXXII. THEOREM.

459. *If two triedral angles have the face angles of one equal respectively to the face angles of the other, their homologous diedral angles are equal.*

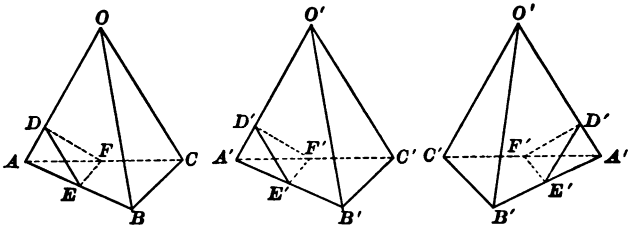


Fig. 1. *Fig. 2.* *Fig. 3.*

Given, in triedral $\angle s$ $O\text{-}ABC$ and $O'\text{-}A'B'C'$,

$$\angle AOB = \angle A'O'B', \ \angle BOC = \angle B'O'C',$$
$$\text{and } \angle COA = \angle C'O'A'.$$

To Prove diedral $\angle OA =$ diedral $\angle O'A'$.

Proof. Lay off OA, OB, OC, $O'A'$, $O'B'$, and $O'C'$ all equal, and draw lines AB, BC, CA, $A'B'$, $B'C'$, and $C'A'$.

$$\therefore \triangle OAB = \triangle O'A'B'. \quad (\S\ 63)$$

$$\therefore AB = A'B'. \quad (\S\ 66)$$

Similarly, $BC = B'C'$ and $CA = C'A'$.

$$\therefore \triangle ABC = \triangle A'B'C'. \quad (\S\ 69)$$

$$\therefore \angle EAF = \angle E'A'F'. \quad (?)$$

On OA and $O'A'$ take $AD = A'D'$.

Draw line DE in face $OAB \perp OA$.

Since $\triangle OAB$ is isosceles, $\angle OAB$ is acute, and hence DE will meet AB; let it meet AB at E.

Also, draw line DF in face $OAC \perp OA$, meeting AC at F; and lines $D'E'$ and $D'F'$ in faces $O'A'B'$ and $O'A'C' \perp O'A'$, meeting $A'B'$ and $A'C'$ at E' and F', respectively.

Draw lines EF and $E'F'$.

Then, in rt. $\triangle s$ ADE and $A'D'E'$,

$$AD = A'D'.$$

And since $\triangle OAB = \triangle O'A'B'$,

$\angle DAE = \angle D'A'E'$. (?)

$\therefore \triangle ADE = \triangle A'D'E'$. (§ 89)

$\therefore AE = A'E'$, and $DE = D'E'$. (?)

Similarly, $AF = A'F'$, and $DF = D'F'$.

Then, in $\triangle$s AEF and $A'E'F'$,

$AE = A'E'$, $AF = A'F'$, and $\angle EAF = \angle E'A'F'$.

$\therefore \triangle AEF = \triangle A'E'F'$. (?)

$\therefore EF = E'F'$. (?)

Then, in $\triangle$s DEF and $D'E'F'$,

$DE = D'E'$, $DF = D'F'$, and $EF = E'F'$.

$\therefore \triangle DEF = \triangle D'E'F'$. (?)

$\therefore \angle EDF = \angle E'D'F'$. (?)

But, EDF and $E'D'F'$ are the plane $\angle$s of diedral $\angle$s OA and $O'A'$, respectively. (§ 429)

$\therefore$ diedral $\angle OA =$ diedral $\angle O'A'$. (§ 433)

Note. The above proof holds for Fig. 3 as well as for Fig. 2; in Figs. 1 and 2, the equal parts occur in the *same* order, and in Figs. 1 and 3 in the *reverse* order.

460. Cor. *If two triedral angles have the face angles of one equal respectively to the face angles of the other,*

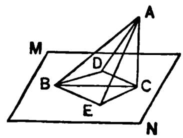
1. *They are equal if the equal parts occur in the same order.*

For if triedral $\angle O'\text{–}A'B'C'$ (Fig. 2) be applied to $O\text{–}ABC$ so that diedral $\angle$s $O'A'$ and OA coincide, point O' falling at O, then since $\angle A'O'C' = \angle AOC$, and $\angle A'O'B' = \angle AOB$, $O'B'$ will coincide with OB, and $O'C'$ with OC.

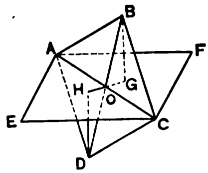
2. *They are symmetrical if the equal parts occur in the reverse order.*

EXERCISES.

18. If BC is the projection of line AB upon plane MN, and BD and BE be drawn in the plane making $\angle CBD = \angle CBE$, prove $\angle ABD = \angle ABE$.



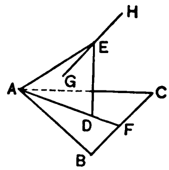
(Lay off $BD = BE$, and draw lines AD, AE, CD, and CE. Prove $\triangle$s ABD and ABE equal.)



19. If a plane be drawn through a diagonal of a parallelogram, the perpendiculars to it from the extremities of the other diagonal are equal.

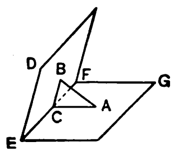
(Given plane *EF* through diagonal *AC* of ▱*ABCD*; to prove *BG* = *DH*. Prove rt. △*BGO* and *DHO* equal.)

20. Two triedral angles are equal when a face angle and the adjacent diedral angles of one are equal respectively to a face angle and the adjacent diedral angles of the other, and similarly placed.

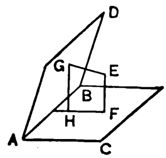


21. *D* is any point in perpendicular *AF* from *A* to side *BC* of triangle *ABC*. If line *DE* be drawn perpendicular to the plane of *ABC*, and line *GH* through *E* parallel to *BC*, prove line *AE* perpendicular to *GH*.

(Prove *BC* ⊥ to plane *AED* by § 438.)

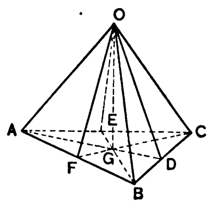


22. *A* is any point in face *EG* of diedral ∠*DEFG*. If *AC* be drawn perpendicular to edge *EF*, and *AB* perpendicular to face *DF*, prove the plane determined by *AC* and *BC* perpendicular to *EF*. (Ex. 9.)



23. From any point *E* within diedral ∠*CABD*, *EF* and *EG* are drawn perpendicular to faces *ABC* and *ABD*, respectively, and *GH* perpendicular to face *ABC* at *H*. Prove *FH* perpendicular to *AB*.

(Prove that *FH* lies in the plane of *EF* and *EG*.)



24. The three planes bisecting the diedral angles of a triedral angle meet in a common straight line.

(Let planes *OAD* and *OBE* intersect in line *OG*. Prove *G* in plane *OCF* by § 444.)

25. Any point in the plane passing through the bisector of an angle, perpendicular to its plane, is equally distant from the sides of the angle.

26. Any face angle of a polyedral angle is less than the sum of the remaining face angles.

(Divide the polyedral ∠ into triedral ∠s by passing planes through any lateral edge.)

Book VII.

POLYEDRONS.

DEFINITIONS.

461. A *polyedron* is a solid bounded by polygons.

The bounding polygons are called the *faces* of the polyedron; their sides are called the *edges*, and their vertices the *vertices*.

A *diagonal* of a polyedron is a straight line joining any two vertices not in the same face.

462. The least number of planes which can form a polyedral angle is three.

Whence, the least number of polygons which can bound a polyedron is four.

A polyedron of four faces is called a *tetraedron;* of six faces, a *hexaedron;* of eight faces, an *octaedron;* of twelve faces, a *dodecaedron;* of twenty faces, an *icosaedron.*

463. A polyedron is called *convex* when the section made by any plane is a convex polygon (§ 121).

All polyedrons considered hereafter will be understood to be convex.

464. The *volume* of a solid is its ratio to another solid, called the *unit of volume,* adopted arbitrarily as the unit of measure (§ 180).

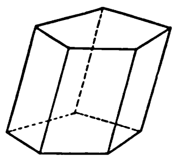
The usual unit of volume is a cube (§ 474) whose edge is some linear unit; for example, a *cubic inch* or a *cubic foot.*

465. Two solids are said to be *equivalent* when their volumes are equal.

PRISMS AND PARALLELOPIPEDS.

DEFINITIONS.

466. A *prism* is a polyedron, two of whose faces are equal polygons lying in parallel planes, having their homologous sides parallel, the other faces being parallelograms (§ 110).



The equal and parallel faces are called the *bases* of the prism, and the other faces the *lateral faces;* the edges which are not sides of the bases are called the *lateral edges,* and the sum of the areas of the lateral faces the *lateral area.*

The *altitude* is the perpendicular distance between the planes of the bases.

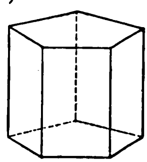
467. The following is given for convenience of reference:

The bases of a prism are equal.

468. It follows from the definition of § 466 that *the lateral edges of a prism are equal and parallel.* (§ 106, I)

469. A prism is called *triangular, quadrangular,* etc., according as its base is a triangle, quadrilateral, etc.

470. A *right prism* is a prism whose lateral edges are perpendicular to its bases.

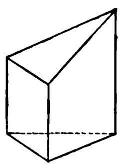


The lateral faces are rectangles (§ 398).

An *oblique prism* is a prism whose lateral edges are not perpendicular to its bases.

471. A *regular prism* is a right prism whose base is a regular polygon.

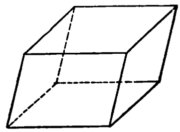
472. A *truncated prism* is a portion of a prism included between the base, and a plane, not parallel to the base, cutting all the lateral edges.



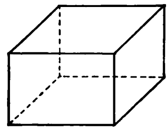
The base of the prism and the section made by the plane are called the *bases* of the truncated prism.

473. A *right section* of a prism is a section made by a plane cutting all the lateral edges, and perpendicular to them.

474. A *parallelopiped* is a prism whose bases are parallelograms; that is, all the faces are parallelograms.



A *right parallelopiped* is a parallelopiped whose lateral edges are perpendicular to its bases.

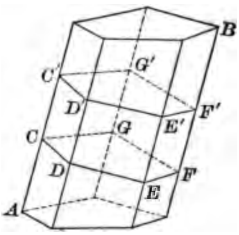


A *rectangular parallelopiped* is a right parallelopiped whose bases are rectangles; that is, all the faces are rectangles.

A *cube* is a rectangular parallelopiped whose six faces are all squares.

PROP. I. THEOREM.

475. *The sections of a prism made by two parallel planes which cut all the lateral edges, are equal polygons.*



Given ∥ planes CF and $C'F'$ cutting all the lateral edges of prism AB.

To Prove section $CDEFG$ = section $C'D'E'F'G'$.

Proof. We have $CD \parallel C'D'$, $DE \parallel D'E'$, etc. (§ 414)

$\therefore$ $CD = C'D'$, $DE = D'E'$, etc. (§ 107)

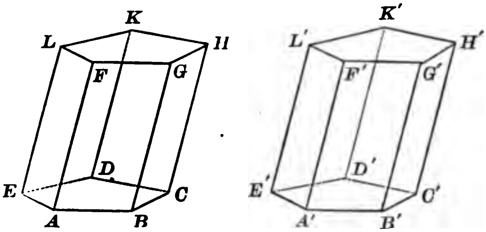
Also $\angle CDE = \angle C'D'E'$, $\angle DEF = \angle D'E'F'$, etc. (§ 426)

Then, polygons $CDEFG$ and $C'D'E'F'G'$, being mutually equilateral and mutually equiangular, are equal. (§ 124)

476. Cor. *The section of a prism made by a plane parallel to the base is equal to the base.*

PROP. II. THEOREM.

477. *Two prisms are equal when the faces including a triedral angle of one are equal respectively to the faces including a triedral angle of the other, and similarly placed.*



Given, in prisms AH and $A'H'$, faces $ABCDE$, AG, and AL equal respectively to faces $A'B'C'D'E'$, $A'G'$, and $A'L'$; the equal parts being similarly placed.

To Prove prism $AH =$ prism $A'H'$.

Proof. We have $\angle s\ EAB$, EAF, and FAB equal respectively to $\angle s\ E'A'B'$, $E'A'F'$, and $F'A'B'$. (§ 66)

$\therefore$ triedral $\angle A\text{-}BEF =$ triedral $\angle A'\text{-}B'E'F'$. (§ 460, 1)

Then, prism $A'H'$ may be applied to prism AH in such a way that vertices A', B', C', D', E', G', F', and L' shall fall at A, B, C, D, E, G, F, and L, respectively.

Now since the lateral edges of the prisms are $\parallel$, edge $C'H'$ will fall on CH, $D'K'$ on DK, etc. (§ 53)

And since points G', F', and L' fall at G, F, and L, respectively, planes LH and $L'H'$ coincide. (§ 395, II)

Then points H' and K' fall at H and K, respectively.

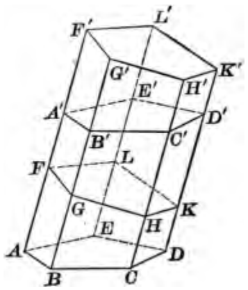
Hence, the prisms coincide throughout, and are equal.

478. Cor. *Two right prisms are equal when they have equal bases and equal altitudes*; for by inverting one of the prisms if necessary, the equal faces will be similarly placed.

479. Sch. The demonstration of § 477 applies without change to the case of two *truncated prisms.*

PROP. III. THEOREM.

480. *An oblique prism is equivalent to a right prism, having for its base a right section of the oblique prism, and for its altitude a lateral edge of the oblique prism.*



Given FK' a right prism, having for its base FK a right section of oblique prism AD', and its altitude FF' equal to AA', a lateral edge of AD'.

To Prove $AD' \Leftrightarrow FK'$.

Proof. In truncated prisms AK and $A'K'$, faces $FGHKL$ and $F'G'H'K'L'$ are equal. (§ 475)

Therefore, $A'K'$ may be applied to AK so that vertices F', G', etc., shall fall at F, G, etc., respectively.

Then, edges $A'F'$, $B'G'$, etc., will coincide in direction with AF, BG, etc., respectively. (§ 399)

But since, by hyp., $FF' = AA'$, we have $AF = A'F'$.

In like manner, $BG = B'G'$, $CH = C'H'$, etc.

Hence, vertices A', B', etc., will fall at A, B, etc., respectively.

Then, $A'K'$ and AK coincide throughout, and are equal.

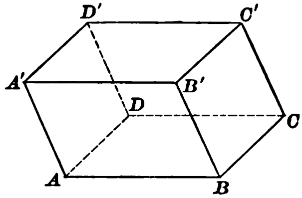
Now taking from the entire solid AK' truncated prism $A'K'$, there remains prism AD'.

And taking its equal AK, there remains prism FK'.

$$\therefore AD' \Leftrightarrow FK'.$$

PROP. IV. THEOREM.

481. *The opposite lateral faces of a parallelopiped are equal and parallel.*



Given AC and $A'C'$ the bases of parallelopiped AC''.

To Prove faces AB' and DC' equal and ∥.

Proof. AB is equal and ∥ to DC, and AA' to DD'. (§ 106, I)

$\therefore \angle A'AB = \angle D'DC$, and $AB' \parallel DC'$. (§ 426)

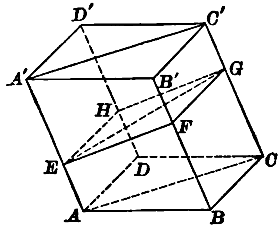
$\therefore$ face AB' = face DC''. (§ 113)

Similarly, we may prove AD' and BC'' equal and ∥.

482. Cor. *Either face of a parallelopiped may be taken as the base.*

PROP. V. THEOREM.

483. *The plane passed through two diagonally opposite edges of a parallelopiped divides it into two equivalent triangular prisms.*



Given plane AC' passing through edges AA' and CC' of parallelopiped $A'C$.

To Prove prism $ABC\text{–}A' \Leftrightarrow$ prism $ACD\text{–}A'$.

Proof. Let $EFGH$ be a right section of the parallelopiped, intersecting plane $AA'C'C$ in line EG.

Now, face $AB' \parallel$ face DC'. (§ 481)

$\therefore EF \parallel GH$. (§ 414)

In like manner, $EH \parallel FG$, and $EFGH$ is a ▱.

$\therefore \triangle EFG = \triangle EGH$. (§ 108)

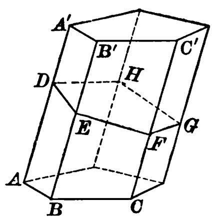
Now, $ABC\text{–}A'$ is ≎ a right prism whose base is EFG and altitude AA', and $ACD\text{–}A'$ is ≎ a right prism whose base is EGH and altitude AA'. (§ 480)

But these right prisms are equal, for they have equal bases and the same altitude. (§ 478)

$\therefore ABC\text{–}A'$ ≎ $ACD\text{–}A'$.

Prop. VI. Theorem.

484. *The lateral area of a prism is equal to the perimeter of a right section multiplied by a lateral edge.*



Given $DEFGH$ a right section of prism AC'.

To Prove lat. area $AC' = (DE + EF + \text{etc.}) \times AA'$.

Proof. We have, $AA' \perp DE$. (§ 398)

$\therefore$ area $AA'B'B = DE \times AA'$. (§ 309)

Similarly, area $BB'C'C = EF \times BB'$

$= EF \times AA'$; etc. (§ 468)

Adding these equations, we have

$$\text{lat. area } AC' = DE \times AA' + EF \times AA' + \text{etc.}$$
$$= (DE + EF + \text{etc.}) \times AA'.$$

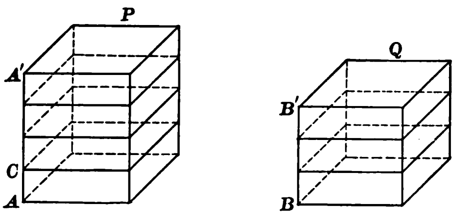
485. Cor. *The lateral area of a right prism is equal to the perimeter of the base multiplied by the altitude.*

PROP. VII. THEOREM.

486. *Two rectangular parallelopipeds having equal bases are to each other as their altitudes.*

Note. The phrase "rectangular parallelopiped" in the above statement signifies the *volume* of the rectangular parallelopiped.

CASE I. *When the altitudes are commensurable.*



Given P and Q rect. parallelopipeds, with equal bases, and commensurable altitudes, AA' and BB'.

To Prove $\frac{P}{Q} = \frac{AA'}{BB'}$.

Proof. Let AC be a common measure of AA' and BB', and suppose it to be contained 4 times in AA', and 3 times in BB'.

$$\therefore \frac{AA'}{BB'} = \frac{4}{3}. \qquad (1)$$

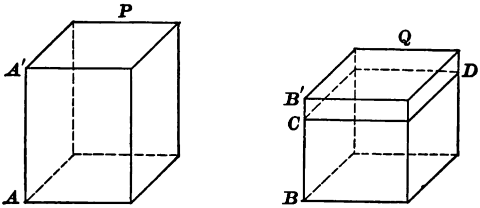
Through the several points of division of AA' and BB' pass planes $\perp$ to lines AA' and BB', respectively.

Then, rect. parallelopiped P will be divided into 4 parts, and rect. parallelopiped Q into 3 parts, all of which parts will be equal. (§ 478)

$$\therefore \frac{P}{Q} = \frac{4}{3}. \qquad (2)$$

From (1) and (2), $\frac{P}{Q} = \frac{AA'}{BB'}$. (?)

CASE II. *When the altitudes are incommensurable.*



Given P and Q rect. parallelopipeds, with equal bases, and incommensurable altitudes, AA' and BB'.

To Prove $$\frac{P}{Q}=\frac{AA'}{BB'}.$$

Proof. Divide AA' into any number of equal parts, and apply one of these parts to BB' as a unit of measure.

Since AA' and BB' are incommensurable, a certain number of the parts will extend from B to C, leaving a remainder $CB' <$ one of the parts.

Draw plane $CD \perp BB'$, and let rect. parallelopiped BD be denoted by Q'.

Then since, by const., AA' and BC are commensurable,

$$\frac{P}{Q'}=\frac{AA'}{BC}. \qquad (\S\ 486, \text{Case I})$$

Now let the number of subdivisions of AA' be indefinitely increased.

Then the length of each part will be indefinitely diminished, and remainder CB' will approach the limit 0.

Then, $\frac{P}{Q'}$ will approach the limit $\frac{P}{Q}$,

and $\frac{AA'}{BC}$ will approach the limit $\frac{AA'}{BB'}$.

$$\therefore \frac{P}{Q}=\frac{AA'}{BB'}. \qquad (\S\ 188)$$

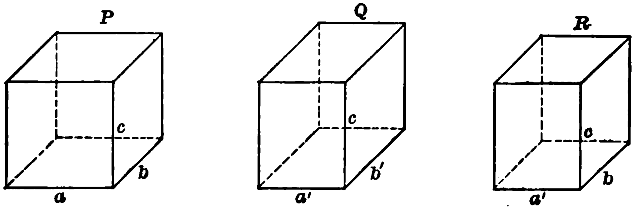
487. Def. The *dimensions* of a rectangular parallelopiped are the three edges which meet at any vertex.

488. Sch. The theorem of § 486 may be expressed:

If two rectangular parallelopipeds have two dimensions of one equal respectively to two dimensions of the other, they are to each other as their third dimensions.

Prop. VIII. Theorem.

489. *Two rectangular parallelopipeds having equal altitudes are to each other as their bases.*



Given P and Q rect. parallelopipeds, with the same altitude c, and the dimensions of the bases a, b, and a', b', respectively.

To Prove $$\frac{P}{Q}=\frac{a\times b}{a'\times b'}. \qquad (§ 305)$$

Proof. Let R be a rect. parallelopiped with the altitude c, and the dimensions of the base a' and b.

Then since P and R have each the dimensions b and c, they are to each other as their third dimensions a and a'. (§ 488)

That is, $$\frac{P}{R}=\frac{a}{a'}. \qquad (1)$$

And since R and Q have each the dimensions a' and c,

$$\frac{R}{Q}=\frac{b}{b'}. \qquad (2)$$

Multiplying (1) and (2), we have

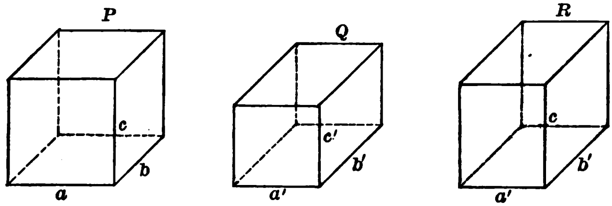
$$\frac{P}{R}\times\frac{R}{Q}, \text{ or } \frac{P}{Q}=\frac{a\times b}{a'\times b'}.$$

490. Sch. The theorem of § 489 may be expressed:

Two rectangular parallelopipeds having a dimension of one equal to a dimension of the other, are to each other as the products of their other two dimensions.

Prop. IX. Theorem.

491. *Any two rectangular parallelopipeds are to each other as the products of their three dimensions.*



Given P and Q rect. parallelopipeds with the dimensions a, b, c, and a', b', c', respectively.

To Prove $$\frac{P}{Q}=\frac{a \times b \times c}{a' \times b' \times c'}.$$

(Let R be a rect. parallelopiped with the dimensions a', b', and c, and find values of $\frac{P}{R}$ and $\frac{R}{Q}$ by §§ 490 and 488.)

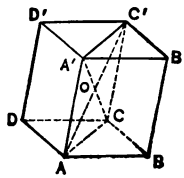
EXERCISES.

1. Two rectangular parallelopipeds, with equal altitudes, have the dimensions of their bases 6 and 14, and 7 and 9, respectively. Find the ratio of their volumes.

2. Find the ratio of the volumes of two rectangular parallelopipeds, whose dimensions are 8, 12, and 21, and 14, 15, and 24, respectively.

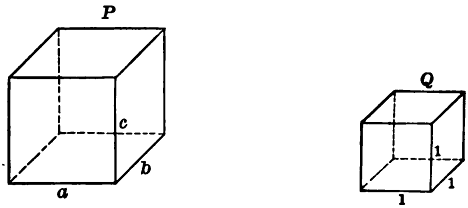
3. The diagonals of a parallelopiped bisect each other.

(To prove that AC' and $A'C$ bisect each other. Prove $AA'C'C$ a ▱ by § 110.)



Prop. X. Theorem.

492. *If the unit of volume is the cube whose edge is the linear unit, the volume of a rectangular parallelopiped is equal to the product of its three dimensions.*



Given a, b, and c the dimensions of rect. parallelopiped P, and Q the unit of volume; that is, a cube whose edge is the linear unit.

To Prove vol. $P = a \times b \times c$.

Proof. We have $\frac{P}{Q} = \frac{a \times b \times c}{1 \times 1 \times 1}$ (§ 491)

$= a \times b \times c$.

But since Q is the unit of volume,

$\frac{P}{Q} =$ vol. P. (§ 464)

$\therefore$ vol. $P = a \times b \times c$.

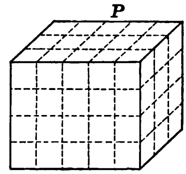
493. Sch. I. In all succeeding theorems relating to volumes, it is understood that the *unit of volume* is the cube whose edge is the linear unit, and the *unit of surface* the square whose side is the linear unit. (Compare § 306.)

494. Cor. I. *The volume of a cube is equal to the cube of its edge.*

495. Cor. II. *The volume of a rectangular parallelopiped is equal to the product of its base and altitude.*

(The proof is left to the pupil.)

496. Sch. II. If the dimensions of the rectangular parallelopiped are *multiples* of the linear unit, the truth of Prop. X. may be seen by dividing the solid into cubes, each equal to the unit of volume.



Thus, if the dimensions of rectangular parallelopiped P are 5 units, 4 units, and 3 units, respectively, the solid can evidently be divided into 60 cubes.

In this case, 60, the number which expresses the volume of the rectangular parallelopiped, is the product of 5, 4, and 3, the numbers which express the lengths of its edges.

EXERCISES.

4. Find the altitude of a rectangular parallelopiped, the dimensions of whose base are 21 and 30, equivalent to a rectangular parallelopiped whose dimensions are 27, 28, and 35.

5. Find the edge of a cube equivalent to a rectangular parallelopiped whose dimensions are 9 in., 1 ft. 9 in., and 4 ft. 1 in.

6. Find the volume, and the area of the entire surface of a cube whose edge is $3\frac{1}{4}$ in.

7. Find the area of the entire surface of a rectangular parallelopiped, the dimensions of whose base are 11 and 13, and volume 858.

8. Find the volume of a rectangular parallelopiped, the dimensions of whose base are 14 and 9, and the area of whose entire surface is 620.

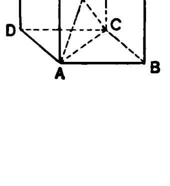
9. Find the dimensions of the base of a rectangular parallelopiped, the area of whose entire surface is 320, volume 336, and altitude 4.

(Represent the dimensions of the base by x and y.)

10. How many bricks, each 8 in. long, $2\frac{3}{4}$ in. wide, and 2 in. thick, will be required to build a wall 18 ft. long, 3 ft. high, and 11 in. thick?

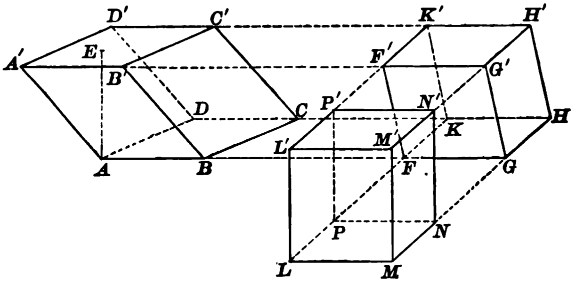
11. The diagonals of a rectangular parallelopiped are equal.

(Prove $AA'C'C$ a rectangle.)



PROP. XI. THEOREM.

497. *The volume of any parallelopiped is equal to the product of its base and altitude.*



Given AE the altitude of parallelopiped AC'.

To Prove vol. $AC' = \text{area } ABCD \times AE$.

Proof. Produce edges AB, $A'B'$, $D'C'$, and DC.

On AB produced, take $FG = AB$; and draw planes FK' and $GH' \perp FG$, forming right parallelopiped FH'.

$\therefore FH' \Leftrightarrow AC'$. (§ 480)

Produce edges HG, $H'G'$, $K'F'$, and KF.

On HG produced, take $NM = HG$; and draw planes NP' and $ML' \perp NM$, forming right parallelopiped LN'.

$\therefore LN' \Leftrightarrow FH'$. (§ 480)

$\therefore LN' \Leftrightarrow AC'$.

Now since, by cons., FG is $\perp$ plane GH', planes LH and MH' are $\perp$. (§ 441)

Then MM', being $\perp MN$, is $\perp$ plane LH. (§ 438)

Whence, $\angle LMM'$ is a rt. $\angle$. (§ 398)

Then, LM' is a rectangle. (§ 76)

Therefore LN' is a rectangular parallelopiped.

$\therefore$ vol. $LN' = \text{area } LMNP \times MM'$. (§ 495)

$\therefore$ vol. $AC' = \text{area } LMNP \times MM'$. (1)

But rect. $LMNP =$ rect. $FGHK$; for they have equal bases MN and GH, and the same altitude. (§ 114)

Also, rect. $FGHK \approx \square ABCD$; for they have equal bases FG and AB, and the same altitude. (§ 310)

$$\therefore LMNP \approx ABCD.$$

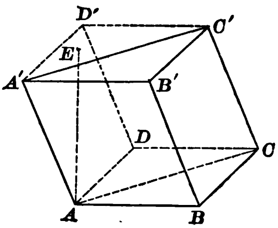
Also, $MM' = AE.$ (§ 424)

Substituting these values in (1), we have

$$\text{vol. } AC' = \text{area } ABCD \times AE.$$

Prop. XII. Theorem.

498. *The volume of a triangular prism is equal to the product of its base and altitude.*



Given AE the altitude of triangular prism $ABC–C'$.

To Prove vol. $ABC–C' =$ area $ABC \times AE$.

Proof. Construct parallelopiped $ABCD–D'$, having its edges ‖ to AB, BC, and BB', respectively.

$\therefore$ vol. $ABC–C' = \frac{1}{2}$ vol. $ABCD–D'$ (§ 483)

$= \frac{1}{2}$ area $ABCD \times AE$ (§ 497)

$=$ area $ABC \times AE$. (§ 108)

EXERCISES.

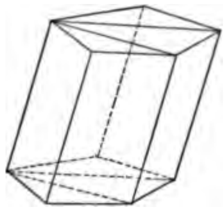
12. Find the lateral area and volume of a regular triangular prism, each side of whose base is 5, and whose altitude is 8.

13. The square of a diagonal of a rectangular parallelopiped is equal to the sum of the squares of its dimensions.

(Fig. of Ex. 11. To prove $\overline{A'C}^2 = \overline{AA'}^2 + \overline{AB}^2 + \overline{AD}^2$.)

PROP. XIII. THEOREM.

499. *The volume of any prism is equal to the product of its base and altitude.*



Given any prism.

To Prove its volume equal to the product of its base and altitude.

Proof. The prism may be divided into triangular prisms by passing planes through one of the lateral edges and the corresponding diagonals of the base.

The volume of each triangular prism is equal to the product of its base and altitude. (§ 498)

Then, the sum of the volumes of the triangular prisms is equal to the sum of their bases multiplied by their common altitude.

Therefore, the volume of the given prism is equal to the product of its base and altitude.

500. Cor. I. *Two prisms having equivalent bases and equal altitudes are equivalent.*

501. Cor. II. 1. *Two prisms having equal altitudes are to each other as their bases.*

2. *Two prisms having equivalent bases are to each other as their altitudes.*

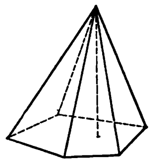
3. *Any two prisms are to each other as the products of their bases by their altitudes.*

Ex. 14. Find the lateral area and volume of a regular hexagonal prism, each side of whose base is 3, and whose altitude is 9.

PYRAMIDS.

DEFINITIONS.

502. A *pyramid* is a polyedron bounded by a polygon, called the *base,* and a series of triangles having a common vertex.



The common vertex of the triangular faces is called the *vertex* of the pyramid.

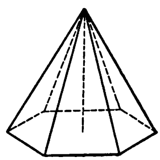
The triangular faces are called the *lateral faces,* and the edges terminating at the vertex the *lateral edges.*

The sum of the areas of the lateral faces is called the *lateral area.*

The *altitude* is the perpendicular distance from the vertex to the plane of the base.

503. A pyramid is called *triangular, quadrangular,* etc., according as its base is a triangle, quadrilateral, etc.

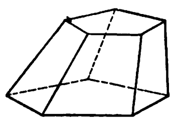
504. A *regular pyramid* is a pyramid whose base is a regular polygon, and whose vertex lies in the perpendicular erected at the centre of the base.



505. A *truncated pyramid* is a portion of a pyramid included between the base and a plane cutting all the lateral edges.

The base of the pyramid and the section made by the plane are called the *bases* of the truncated pyramid.

506. A *frustum of a pyramid* is a truncated pyramid whose bases are parallel.



The *altitude* is the perpendicular distance between the planes of the bases.

EXERCISES.

15. Find the length of the diagonal of a rectangular parallelopiped whose dimensions are 8, 9, and 12.

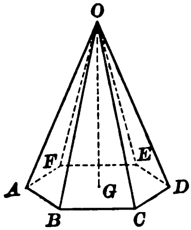
16. The diagonal of a cube is equal to its edge multiplied by $\sqrt{3}$.

Prop. XIV. Theorem.

507. *In a regular pyramid,*

I. *The lateral edges are equal.*

II. *The lateral faces are equal isosceles triangles.*



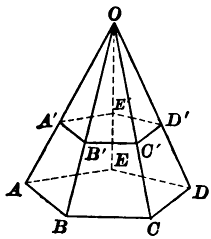
(The theorem follows by §§ 406, I, and 69.)

508. Def. The *slant height* of a regular pyramid is the altitude of any lateral face.

Or, it is the line drawn from the vertex of the pyramid to the middle point of any side of the base. (§ 94, I)

Prop. XV. Theorem.

509. *The lateral faces of a frustum of a regular pyramid are equal trapezoids.*



Given AC' a frustum of regular pyramid $O\text{-}ABCDE$.

To Prove faces AB' and BC' equal trapezoids.

Proof. We have $\triangle OAB = \triangle OBC$. (§ 507, II)

We may then apply $\triangle OAB$ to $\triangle OBC$ in such a way that sides OB, OA, and AB shall coincide with sides OB, OC, and BC, respectively.

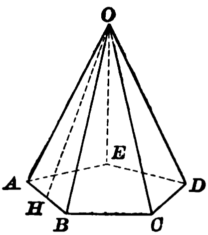
Now, $A'B' \parallel AB$ and $B'C' \parallel BC$. (?)
Hence, line $A'B'$ will coincide with line $B'C'$. (§ 53)
Then, AB' and BC' coincide throughout, and are equal.

510. Cor. *The lateral edges of a frustum of a regular pyramid are equal.*

511. Def. The *slant height* of a frustum of a regular pyramid is the altitude of any lateral face.

PROP. XVI. THEOREM.

512. ***The lateral area of a regular pyramid is equal to the perimeter of its base multiplied by one-half its slant height.***



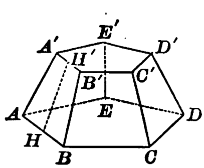
Given slant height OH of regular pyramid $O\text{–}ABCDE$.

To Prove

lat. area $O\text{–}ABCDE = (AB + BC + \text{etc.}) \times \frac{1}{2} OH$.

(By § 508, OH is the altitude of each lateral face.)

513. Cor. *The lateral area of a frustum of a regular pyramid is equal to one-half the sum of the perimeters of its bases, multiplied by its slant height.*



Given slant height HH' of the frustum of a regular pyramid AD'.

To Prove

lat. area $AD' = \frac{1}{2}(AB + A'B' + BC + B'C' + \text{etc.}) \times HH'$.

(HH' is the altitude of each lateral face.)

EXERCISES.

17. The volume of a cube is $4\frac{17}{27}$ cu. ft. Find the area of its entire surface in square inches.

18. The volume of a right prism is 2310, and its base is a right triangle whose legs are 20 and 21, respectively. Find its lateral area.

19. Find the lateral area and volume of a right triangular prism, having the sides of its base 4, 7, and 9, respectively, and the altitude 8.

20. The volume of a regular triangular prism is $96\sqrt{3}$, and one side of its base is 8. Find its lateral area.

21. The diagonal of a cube is $8\sqrt{3}$. Find its volume, and the area of its entire surface.

(Represent the edge by x.)

22. A trench is 124 ft. long, $2\frac{1}{2}$ ft. deep, 6 ft. wide at the top, and 5 ft. wide at the bottom. How many cubic feet of water will it contain? (§§ 316, 409.)

23. The lateral area and volume of a regular hexagonal prism are 60 and $15\sqrt{3}$, respectively. Find its altitude, and one side of its base.

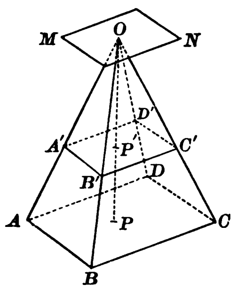
(Represent the altitude by x, and the side of the base by y.)

PROP. XVII. THEOREM.

514. *If a pyramid be cut by a plane parallel to its base,*

I. *The lateral edges and the altitude are divided proportionally.*

II. *The section is similar to the base.*



Given plane $A'C'$ ∥ to base of pyramid $O\text{–}ABCD$, cutting faces OAB, OBC, OCD, and ODA in lines $A'B'$, $B'C'$, $C'D'$, and $D'A'$, respectively, and altitude OP at P'.

I. **To Prove** $\frac{OA'}{OA} = \frac{OB'}{OB} = \frac{OC'}{OC}$ etc. $= \frac{OP'}{OP}$.

Proof. Through O pass plane $MN \parallel ABCD$.

$$\therefore \frac{OA'}{OA} = \frac{OB'}{OB} = \frac{OC'}{OC} \text{ etc.} = \frac{OP'}{OP}. \qquad (\S\ 427)$$

II. **To Prove** section $A'B'C'D'$ similar to $ABCD$.

Proof. We have $A'B' \parallel AB$, $B'C' \parallel BC$, etc. (?)

$\therefore \angle A'B'C' = \angle ABC$, $\angle B'C'D' = \angle BCD$, etc. (§ 426)

Again, $\triangle$ $OA'B'$, $OB'C'$, etc., are similar to $\triangle$ OAB, OBC, etc., respectively. (§ 257)

$$\therefore \frac{OA'}{OA} = \frac{A'B'}{AB}, \ \frac{OB'}{OB} = \frac{B'C'}{BC} \text{ etc.} \qquad (1)$$

But, $$\frac{OA'}{OA} = \frac{OB'}{OB} \text{ etc.} \qquad (\S\ 514, \text{I})$$

$$\therefore \frac{A'B'}{AB} = \frac{B'C'}{BC} = \frac{C'D'}{CD} \text{ etc.} \qquad (?)$$

Then, polygons $A'B'C'D'$ and $ABCD$ are mutually equiangular, and have their homologous sides proportional.

Whence, $A'B'C'D'$ and $ABCD$ are similar. (§ 252)

515. Cor. I. Since $A'B'C'D'$ and $ABCD$ are similar,

$$\frac{\text{area } A'B'C'D'}{\text{area } ABCD} = \frac{\overline{A'B'}^2}{\overline{AB}^2}. \qquad (\S\ 322)$$

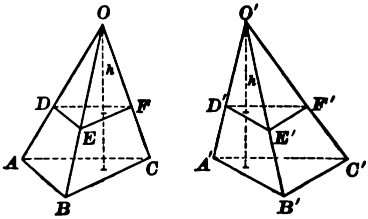
But from (1), § 514, $$\frac{A'B'}{AB} = \frac{OA'}{OA}$$

$$= \frac{OP'}{OP}. \qquad (\S\ 514, \text{I})$$

$$\therefore \frac{\text{area } A'B'C'D'}{\text{area } ABCD} = \frac{\overline{OP'}^2}{\overline{OP}^2}.$$

Hence, *the area of a section of a pyramid, parallel to the base, is to the area of the base as the square of its distance from the vertex is to the square of the altitude of the pyramid.*

516. Cor. II. *If two pyramids have equal altitudes and equivalent bases, sections parallel to the bases equally distant from the vertices are equivalent.*



Given bases of pyramids $O\text{-}ABC$ and $O'\text{-}A'B'C'$ ⇔, and the altitude of each pyramid $= H$; also DEF and $D'E'F'$ sections ∥ to the bases at distance h from O and O', respectively.

To Prove area $DEF =$ area $D'E'F'$.

Proof. We have

$$\frac{\text{area } DEF}{\text{area } ABC} = \frac{h^2}{H^2}, \text{ and } \frac{\text{area } D'E'F'}{\text{area } A'B'C'} = \frac{h^2}{H^2}. \quad (\S\ 515)$$

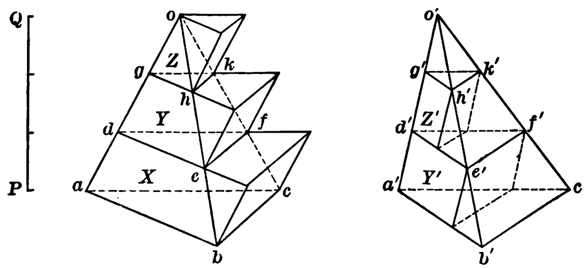
$$\therefore \frac{\text{area } DEF}{\text{area } ABC} = \frac{\text{area } D'E'F'}{\text{area } A'B'C'}. \quad (?)$$

But by hyp., area $ABC =$ area $A'B'C'$.

$\therefore$ area $DEF =$ area $D'E'F'$.

Prop. XVIII. Theorem.

517. *Two triangular pyramids having equal altitudes and equivalent bases are equivalent.*



Given $o\text{-}abc$ and $o'\text{-}a'b'c'$ triangular pyramids with equal altitudes and ⇔ bases.

To Prove vol. $o\text{–}abc =$ vol. $o'\text{–}a'b'c'$.

Proof. Place the pyramids with their bases in the same plane, and let PQ be their common altitude.

Divide PQ into any number of equal parts.

Through the points of division pass planes ∥ to the plane of the bases, cutting $o\text{–}abc$ in sections def and ghk, and $o'\text{–}a'b'c'$ in sections $d'e'f'$ and $g'h'k'$, respectively.

$$\therefore\ def \Leftrightarrow d'e'f', \text{ and } ghk \Leftrightarrow g'h'k'. \qquad (\S\ 516)$$

With abc, def, and ghk as *lower* bases, construct prisms X, Y, and Z, with their lateral edges equal and ∥ to ad; and with $d'e'f'$ and $g'h'k'$ as *upper* bases, construct prisms Y' and Z', with their lateral edges equal and ∥ to $a'd'$.

$$\therefore \text{ prism } Y \Leftrightarrow \text{prism } Y', \text{ and prism } Z \Leftrightarrow \text{prism } Z'. \quad (\S\ 500)$$

Hence, the sum of the prisms circumscribed about $o\text{–}abc$ exceeds the sum of the prisms inscribed in $o'\text{–}a'b'c'$ by prism X.

But, $o\text{–}abc$ is evidently $<$ the sum of prisms X, Y, and Z; and it is $>$ the sum of prisms $\Leftrightarrow$ to Y' and Z', respectively, which can be constructed with def and ghk as *upper* bases, having their lateral edges equal and ∥ to ad.

Again, $o'\text{–}a'b'c'$ is $>$ the sum of prisms Y' and Z'; and it is $<$ the sum of prisms $\Leftrightarrow$ to X, Y, and Z, respectively, which can be constructed with $a'b'c'$, $d'e'f'$, and $g'h'k'$ as *lower* bases, having their lateral edges equal and ∥ to $a'd'$.

That is, each pyramid is $<$ the sum of prisms X, Y, and Z, and $>$ the sum of prisms Y' and Z'; whence, the difference of the volumes of the pyramids must be $<$ the difference of the volumes of the two systems of prisms, or $<$ volume X.

Now by sufficiently increasing the number of subdivisions of PQ, the volume of prism X may be made $<$ any assigned volume, however small.

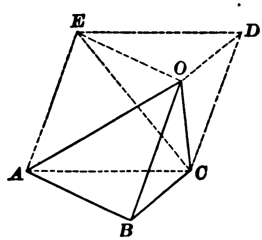
Hence, the volumes of the pyramids cannot differ by any volume, however small.

$$\therefore \text{ vol. } o\text{–}abc = \text{vol. } o'\text{–}a'b'c'.$$

518. Cor. Since vol. $o'-a'b'c'$ is > the total volume of the inscribed prisms, and < the total volume of the circumscribed, the difference between vol. $o'-a'b'c'$ and the total volume of the inscribed prisms is < the difference between the total volumes of the two systems of prisms, or < vol. X; and hence approaches the limit 0 when the number of subdivisions is indefinitely increased.

Prop. XIX. Theorem.

519. *A triangular pyramid is equivalent to one-third of a triangular prism having the same base and altitude.*



Given triangular pyramid $O-ABC$, and triangular prism $ABC-ODE$ having the same base and altitude.

To Prove vol. $O-ABC = \frac{1}{3}$ vol. $ABC-ODE$.

Proof. Prism $ABC-ODE$ is composed of triangular pyramid $O-ABC$, and quadrangular pyramid $O-ACDE$.

Divide the latter into two triangular pyramids, $O-ACE$ and $O-CDE$, by passing a plane through O, C, and E.

Now, $O-ACE$ and $O-CDE$ have the same altitude.

And since CE is a diagonal of ▱ $ACDE$, they have equal bases, ACE and CDE. (§ 108)

$\therefore$ vol. $O-ACE =$ vol. $O-CDE$. (§ 517)

Again, pyramid $O-CDE$ may be regarded as having its vertex at C, and △ ODE for its base.

Then, pyramids $O-ABC$ and $C-ODE$ have the same altitude. (§ 424)

They have also equal bases, ABC and ODE. (§ 467)

$\therefore$ vol. $O\text{–}ABC =$ vol. $C\text{–}ODE$. (?)

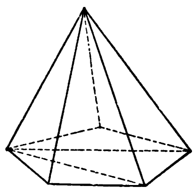
Then, vol. $O\text{–}ABC =$ vol. $O\text{–}ACE =$ vol. $O\text{–}CDE$. (?)

$\therefore$ vol. $O\text{–}ABC = \frac{1}{3}$ vol. $ABC\text{–}ODE$.

520. Cor. *The volume of a triangular pyramid is equal to one-third the product of its base and altitude.* (§ 498)

Prop. XX. Theorem.

521. *The volume of any pyramid is equal to one-third the product of its base and altitude.*



(Prove as in § 499.)

522. Cor. 1. *Two pyramids having equivalent bases and equal altitudes are equivalent.*

2. *Two pyramids having equal altitudes are to each other as their bases.*

3. *Two pyramids having equivalent bases are to each other as their altitudes.*

4. *Any two pyramids are to each other as the products of their bases by their altitudes.*

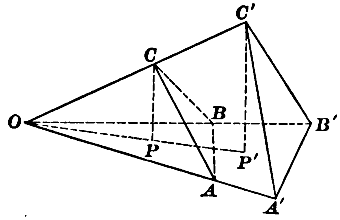
EXERCISES.

24. The altitude of a pyramid is 12 in., and its base is a square 9 in. on a side. What is the area of a section parallel to the base, whose distance from the vertex is 8 in.? (§ 515.)

25. The altitude of a pyramid is 20 in., and its base is a rectangle whose dimensions are 10 in. and 15 in., respectively. What is the distance from the vertex of a section parallel to the base, whose area is 54 sq. in.?

PROP. XXI. THEOREM.

523. *Two tetraedrons having a triedral angle of one equal to a triedral angle of the other, are to each other as the products of the edges including the equal triedral angles.*



Given V and V' the volumes of tetraedrons $O\text{–}ABC$ and $O\text{–}A'B'C'$, respectively, having the common triedral $\angle\ O$.

To Prove $$\frac{V}{V'}=\frac{OA \times OB \times OC}{OA' \times OB' \times OC'}.$$

Proof. Draw lines CP and $C'P' \perp$ to face $OA'B'$.

Let their plane intersect face $OA'B'$ in line OPP'.

Now, OAB and $OA'B'$ are the bases, and CP and $C'P'$ the altitudes, of triangular pyramids $C\text{–}OAB$ and $C'\text{–}OA'B'$, respectively.

$$\therefore \frac{V}{V'}=\frac{\text{area } OAB \times CP}{\text{area } OA'B' \times C'P'} \qquad (§ 522, 4)$$

$$=\frac{\text{area } OAB}{\text{area } OA'B'} \times \frac{CP}{C'P'}. \qquad (1)$$

But, $$\frac{\text{area } OAB}{\text{area } OA'B'}=\frac{OA \times OB}{OA' \times OB'}. \qquad (§ 321)$$

Also, $\triangle$ OCP and $OC'P'$ are rt. $\triangle$. (§ 398)

Then, $\triangle$ OCP and $OC'P'$ are similar. (§ 256)

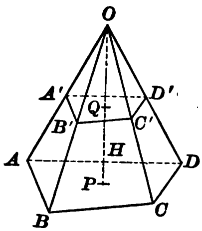
$$\therefore \frac{CP}{C'P'}=\frac{OC}{OC'}. \qquad (?)$$

Substituting these values in (1), we have

$$\frac{V}{V'}=\frac{OA \times OB}{OA' \times OB'} \times \frac{OC}{OC'}=\frac{OA \times OB \times OC}{OA' \times OB' \times OC'}.$$

Prop. XXII. Theorem.

524. *The volume of a frustum of a pyramid is equal to the sum of its bases and a mean proportional between its bases, multiplied by one-third its altitude.*



Given B the area of the lower base, b the area of the upper base, and H the altitude, of AC', a frustum of any pyramid $O\text{–}AC$.

To Prove vol. $AC' = (B + b + \sqrt{B \times b}) \times \frac{1}{3}H$. (§ 233)

Proof. Draw altitude OP, cutting $A'C'$ at Q.

Now, vol. $AC' =$ vol. $O\text{–}AC -$ vol. $O\text{–}A'C'$

$$= B \times \tfrac{1}{3}(H + OQ) - b \times \tfrac{1}{3}OQ \qquad (§ 521)$$

$$= B \times \tfrac{1}{3}H + B \times \tfrac{1}{3}OQ - b \times \tfrac{1}{3}OQ$$

$$= B \times \tfrac{1}{3}H + (B - b) \times \tfrac{1}{3}OQ. \qquad (1)$$

But, $B : b = \overline{OP}^2 : \overline{OQ}^2$. (§ 515)

Taking the square root of each term,

$$\sqrt{B} : \sqrt{b} = OP : OQ. \qquad (§ 241)$$

$$\therefore \sqrt{B} - \sqrt{b} : \sqrt{b} = OP - OQ : OQ \qquad (§ 238)$$

$$= H : OQ.$$

$$\therefore (\sqrt{B} - \sqrt{b}) \times OQ = \sqrt{b} \times H. \qquad (§ 232)$$

Multiplying both members by $(\sqrt{B} + \sqrt{b})$,

$$(B - b) \times OQ = (\sqrt{B \times b} + b) \times H.$$

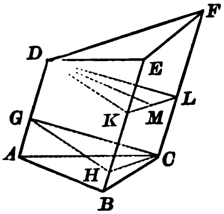
Substituting this value in (1), we have

$$\text{vol. } AC' = B \times \tfrac{1}{3}H + (\sqrt{B \times b} + b) \times \tfrac{1}{3}H$$

$$= (B + b + \sqrt{B \times b}) \times \tfrac{1}{3}H.$$

PROP. XXIII. THEOREM.

525. *The volume of a truncated triangular prism is equal to the product of a right section by one-third the sum of the lateral edges.*



Given GHC and DKL rt. sections of truncated triangular prism $ABC\text{–}DEF$.

To Prove

vol. $ABC\text{–}DEF = \text{area } GHC \times \frac{1}{3}(AD + BE + CF)$.

Proof. Draw line $DM \perp KL$.

The given truncated prism consists of the rt. triangular prism $GHC\text{–}DKL$, and pyramids $D\text{–}EKLF$ and $C\text{–}ABHG$.

vol. $GHC\text{–}DKL = \text{area } GHC \times GD$ (§ 498)

$= \text{area } GHC \times \frac{1}{3}(GD + HK + CL)$, (1)

since the lateral edges of a prism are equal (§ 468).

Now DM is the altitude of pyramid $D\text{–}EKLF$. (§ 438)

$\therefore$ vol. $D\text{–}EKLF = \text{area } EKLF \times \frac{1}{3} DM$. (§ 521)

But KL is the altitude of trapezoid $EKLF$. (§ 398)

$\therefore$ vol. $D\text{–}EKLF = \frac{1}{2}(KE + LF) \times KL \times \frac{1}{3} DM$. (§ 316)

Rearranging the factors, we have

vol. $D\text{–}EKLF = (\frac{1}{2} KL \times DM) \times \frac{1}{3}(KE + LF)$

$= \text{area } DKL \times \frac{1}{3}(KE + LF)$ (§ 312)

$= \text{area } GHC \times \frac{1}{3}(KE + LF)$. (2)

In like manner, we may prove

vol. $C\text{–}ABHG = \text{area } GHC \times \frac{1}{3}(AG + BH)$. (3)

Adding (1), (2), and (3), the sum of the volumes of the solids $GHC\text{–}DKL$, $D\text{–}EKLF$, and $C\text{–}ABHG$ is

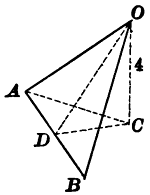
area $GHC \times \frac{1}{3}(\overline{AG + GD} + \overline{BH + HK + KE} + \overline{CL + LF})$.

$\therefore$ vol. $ABC\text{–}DEF = \text{area } GHC \times \frac{1}{3}(AD + BE + CF)$.

526. Cor. *The volume of a truncated right triangular prism is equal to the product of its base by one-third the sum of the lateral edges.*

EXERCISES.

26. Each side of the base of a regular triangular pyramid is 6, and its altitude is 4. Find its lateral edge, lateral area, and volume.



Let OAB be a lateral face of the regular triangular pyramid, and C the centre of the base; draw line $CD \perp AB$; also, lines OC, AC, and OD.

Now, $AC = \frac{AB}{\sqrt{3}}$ (§ 356) $= \frac{6}{\sqrt{3}} = 2\sqrt{3}$.

$\therefore$ lat. edge $OA = \sqrt{\overline{AC}^2 + \overline{OC}^2}$ (§ 272) $= \sqrt{12 + 16} = \sqrt{28} = 2\sqrt{7}$.

$\therefore$ slant ht. $OD = \sqrt{\overline{OA}^2 - \overline{AD}^2}$ (§ 273) $= \sqrt{28 - 9} = \sqrt{19}$.

$\therefore$ lat. area of pyramid $= 9\sqrt{19}$ (§ 512).

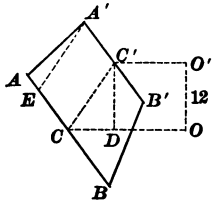
Again, $CD = \sqrt{\overline{AC}^2 - \overline{AD}^2} = \sqrt{12 - 9} = \sqrt{3}$.

$\therefore$ area of base $= \frac{1}{2} \times 18 \times \sqrt{3}$ (§ 350) $= 9\sqrt{3}$.

$\therefore$ vol. of pyramid $= \frac{1}{3} \times 9\sqrt{3} \times 4$ (§ 520) $= 12\sqrt{3}$.

27. Find the lateral edge, lateral area, and volume of a frustum of a regular quadrangular pyramid, the sides of whose bases are 17 and 7, respectively, and whose altitude is 12.

Let $ABB'A'$ be a lateral face of the frustum, and O and O' the centres of the bases; draw lines $OC \perp AB$, $O'C' \perp A'B'$, $C'D \perp OC$, and $A'E \perp AB$; also, lines OO' and CC'.



Now, $CD = OC - O'C' = 8\frac{1}{2} - 3\frac{1}{2} = 5$.

$\therefore$ Slant ht. CC'

$= \sqrt{\overline{CD}^2 + \overline{C'D}^2} = \sqrt{25 + 144} = \sqrt{169} = 13$.

$\therefore$ lat. area frustum

$= \frac{1}{2}(68 + 28) \times 13$ (§ 513) $= 624$.

Again, $AE = AC - A'C' = 8\frac{1}{2} - 3\frac{1}{2} = 5$, and $A'E = CC' = 13$.

$\therefore$ lat. edge $AA' = \sqrt{\overline{AE}^2 + \overline{A'E}^2} = \sqrt{25 + 169} = \sqrt{194}$.

Again, area lower base $= 17^2$, area upper base $= 7^2$, and a mean proportional between them $= \sqrt{17^2 \times 7^2} = 17 \times 7 = 119$.

$\therefore$ vol. frustum $= (289 + 49 + 119) \times 4$ (§ 524) $= 1828$.

Find the lateral edge, lateral area, and volume

28. Of a regular triangular pyramid, each side of whose base is 12, and whose altitude is 15.

29. Of a regular quadrangular pyramid, each side of whose base is 3, and whose altitude is 5.

30. Of a regular hexagonal pyramid, each side of whose base is 4, and whose altitude is 9.

31. Of a frustum of a regular triangular pyramid, the sides of whose bases are 18 and 6, respectively, and whose altitude is 24.

32. Of a frustum of a regular quadrangular pyramid, the sides of whose bases are 9 and 5, respectively, and whose altitude is 10.

33. Of a frustum of a regular hexagonal pyramid, the sides of whose bases are 8 and 4, respectively, and whose altitude is 12.

34. Find the volume of a truncated right triangular prism, the sides of whose base are 5, 12, and 13, and whose lateral edges are 3, 7, and 5, respectively.

35. Find the volume of a truncated regular quadrangular prism, a side of whose base is 8, and whose lateral edges, taken in order, are 2, 6, 8, and 4, respectively.

(Pass a plane through two diagonally opposite lateral edges, dividing the solid into two truncated right triangular prisms.)

36. Find the volume of a truncated right triangular prism, whose lateral edges are 11, 14, and 17, having for its base an isosceles triangle whose sides are 10, 13, and 13, respectively.

37. The slant height and lateral edge of a regular quadrangular pyramid are 25 and $\sqrt{674}$, respectively. Find its lateral area and volume.

38. The altitude and slant height of a regular hexagonal pyramid are 15 and 17, respectively. Find its lateral edge and volume.

(Represent the side of the base by x.)

39. The lateral edge of a frustum of a regular hexagonal pyramid is 10, and the sides of its bases are 10 and 4, respectively. Find its lateral area and volume.

40. Find the lateral area and volume of a frustum of a regular triangular pyramid, the sides of whose bases are 12 and 6, respectively, and whose lateral edge is 5.

41. Find the lateral area and volume of a regular quadrangular pyramid, the area of whose base is 100, and whose lateral edge is 13.

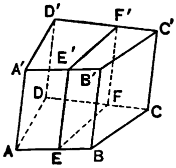
42. Prove the lateral surface of a pyramid greater than its base, when the perpendicular from the vertex to the base falls within the base.

(From foot of altitude draw lines to the vertices of the base; each △ formed has a smaller altitude than the corresponding lateral face.)

43. If E, F, G, and H are the middle points of edges AB, AD, CD, and BC, respectively, of tetraedron $ABCD$, prove $EFGH$ a parallelogram. (§ 130.)

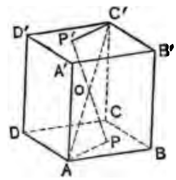
44. Two tetraedrons are equal if a diedral angle and the adjacent faces of one are equal, respectively, to a diedral angle and the adjacent faces of the other, if the equal parts are similarly placed.

(Figs. of § 459. Given faces OAB, OAC, and diedral $\angle OA$ equal, respectively, to faces $O'AB'$, $O'A'C'$, and diedral $\angle O'A'$.)



45. The section of a prism made by a plane parallel to a lateral edge is a parallelogram.

(Given section $EE'F'F \parallel AA'$. Prove $EE' \parallel$ to plane CD'; then use § 412.)



46. The point of intersection of the diagonals of a parallelopiped is called the *centre* of the parallelopiped. (Ex. 3.)

Prove that any line drawn through the centre of a parallelopiped, terminating in a pair of opposite faces, is bisected at that point.

47. The volume of a regular prism is equal to its lateral area, multiplied by one-half the apothem of its base. (§ 350.)

48. The volume of a regular pyramid is equal to its lateral area, multiplied by one-third the distance from the centre of its base to any lateral face.

(Pass planes through the lateral edges and the centre of the base.)

49. Find the area of the entire surface and the volume of a triangular pyramid, each of whose edges is 2.

50. The areas of the bases of a frustum of a pyramid are 12 and 75, respectively, and its altitude is 9. What is the altitude of the pyramid?

(Let altitude of pyramid $= x$; then $x - 9$ is the $\perp$ from its vertex to the upper base of the frustum; then use § 515.)

51. The bases of a frustum of a pyramid are rectangles, whose sides are 27 and 15, and 9 and 5, respectively, and the line joining their centres is perpendicular to each base. If the altitude of the frustum is 12, find its lateral area and volume.

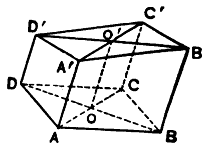
(From the centre of each base draw ⊥s to two of its sides; in this way the altitudes of the lateral faces may be found.)

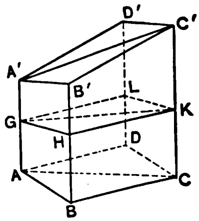
52. A frustum of any pyramid is equivalent to the sum of three pyramids, having for their common altitude the altitude of the frustum, and for their bases the lower base, the upper base, and a mean proportional between the bases, of the frustum. (§ 524.)

53. The upper base of a truncated parallelopiped is a parallelogram.

54. The sum of two opposite lateral edges of a truncated parallelopiped is equal to the sum of the other two lateral edges.

(Let planes AC' and BD' intersect in OO'. Find the length of OO' in terms of the lateral edges by § 132.)





55. The volume of a truncated parallelopiped is equal to the area of a right section, multiplied by one-fourth the sum of the lateral edges.

(By proof of § 483, a rt. section of a parallelopiped is a ▱; divide the solid into two truncated triangular prisms, and apply Ex. 54.)

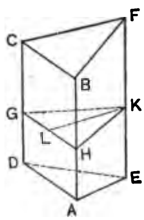
56. The volume of a truncated parallelopiped is equal to the area of a right section, multiplied by the distance between the centres of the bases.

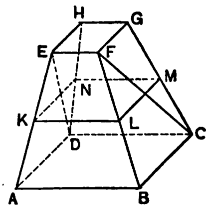
(By Ex. 54, the distance between the centres of the bases may be proved equal to one-fourth the sum of the lateral edges.)

57. If $ABCD$ is a rectangle, and EF any line not in its plane parallel to AB, the volume of the solid bounded by figures $ABCD$, $ABFE$, $CDEF$, ADE, and BCF, is

$$\tfrac{1}{6}h \times AD \times (2\,AB + EF),$$

where h is the perpendicular from any point of EF to $ABCD$. (§ 525.)





58. If $ABCD$ and $EFGH$ are rectangles lying in parallel planes, AB and BC being parallel to EF and FG, respectively, the solid bounded by the figures $ABCD$, $EFGH$, $ABFE$, $BCGF$, $CDHG$, and $DAEH$, is called a *rectangular prismoid.*

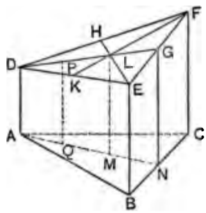
$ABCD$ and $EFGH$ are called the *bases* of the rectangular prismoid, and the perpendicular distance between them the *altitude.*

Prove the volume of a rectangular prismoid equal to the sum of its bases, plus four times a section equally distant from the bases, multiplied by one-sixth the altitude.

(Pass a plane through CD and EF, and find volumes of solids $ABCD$-EF and $EFGH$-CD by Ex. 57.)

59. Find the volume of rectangular prismoid the sides of whose bases are 10 and 7, and 6 and 5, respectively, and whose altitude is 9.

60. Two tetraedrons are equal if three faces of one are equal, respectively, to three faces of the other, if the equal parts are similarly placed. (§ 460, 1.)



61. The perpendicular drawn to the lower base of a truncated right triangular prism from the intersection of the medians of the upper base, is equal to one-third the sum of the lateral edges.

(Let P be the middle point of DL, and draw $PQ \perp ABC$; express LM in terms of PQ and GN by § 132.)

62. The three planes passing through the lateral edges of a triangular pyramid, bisecting the sides of the base, meet in a common straight line.

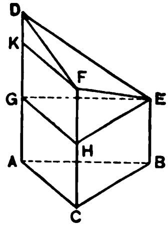
(Fig. of Ex. 24, p. 272. The intersections of the planes with the base of the pyramid are the medians of the base.)

63. A monument is in the form of a frustum of a regular quadrangular pyramid 8 ft. in height, the sides of whose bases are 3 ft. and 2 ft., respectively, surmounted by a regular quadrangular pyramid 2 ft. in height, each side of whose base is 2 ft. What is its weight, at 180 lb. to the cubic foot?

64. Find the area of the base of a regular quadrangular pyramid, whose lateral faces are equilateral triangles, and whose altitude is 5.

(Represent lateral edge and side of base by x.)

65. A plane passed through the centre of a parallelopiped divides it into two equivalent solids. (Ex. 55.)

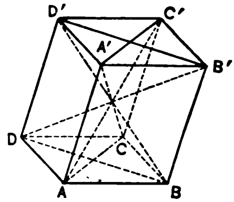


66. The sides of the base, AB, BC, and CA, of truncated right triangular prism $ABC\text{-}DEF$ are 15, 4, and 12, respectively, and the lateral edges AD, BE, and CF are 15, 7, and 10, respectively. Find the area of upper base DEF.

(Draw $EH \perp CF$, and HG and $FK \perp AD$. Find area DEF by § 324.)

67. The volume of a triangular prism is equal to a lateral face, multiplied by one-half its perpendicular distance from any point in the opposite lateral edge.

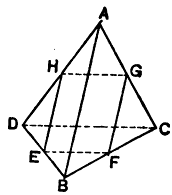
(Draw a rt. section of the prism, and apply § 525.)



68. The sum of the squares of the four diagonals of a parallelopiped is equal to the sum of the squares of its twelve edges.

(To prove $\overline{AC'}^2 + \overline{A'C}^2 + \overline{BD'}^2 + \overline{B'D}^2$ equal to $4\,\overline{AA'}^2 + 4\,\overline{AB}^2 + 4\,\overline{AD}^2$. Apply Ex. 79, p. 228, to $\square AA'C'C$.)

69. The altitude and lateral edge of a frustum of a regular triangular pyramid are 8 and 10, respectively, and each side of its upper base is $2\sqrt{3}$. Find its volume and lateral area.



70. If $ABCD$ is a tetraedron, the section made by a plane parallel to each of the edges AB and CD is a parallelogram. (§ 412.)

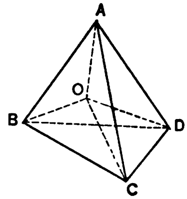
(To prove $EFGH$ a $\square$.)

71. In tetraedron $ABCD$, a plane is drawn through edge CD perpendicular to AB, intersecting faces ABC and ABD in CE and ED, respectively. If the bisector of $\angle CED$ meets CD at F, prove

$$CF : DF = \text{area } ABC : \text{area } ABD. \quad (\S\ 249.)$$

72. The sum of the perpendiculars drawn to the faces from any point within a regular tetraedron (§ 536) is equal to its altitude.

(Divide the tetraedron into triangular pyramids, having the given point for their common vertex.)



73. The planes bisecting the diedral angles of a tetraedron intersect in a common point.

74. If the four diagonals of a quadrangular prism pass through a common point, the prism is a parallelopiped.

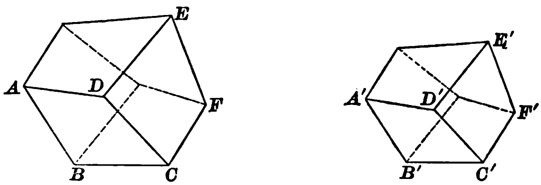
(In Fig. of Ex. 68, let AC', $A'C$, BD', and $B'D$ pass through a common point. To prove AC' a parallelopiped. Prove AC a ▱.)

SIMILAR POLYEDRONS.

527. Def. Two polyedrons are said to be *similar* when they have the same number of faces similar each to each and similarly placed, and have their homologous polyedral angles equal.

PROP. XXIV. THEOREM.

528. *The ratio of any two homologous edges of two similar polyedrons is equal to the ratio of any other two homologous edges.*



Given, in similar polyedrons AF and $A'F'$, edge AB homologous to edge $A'B'$, and edge EF to edge $E'F'$; and faces AC and DF similar to faces $A'C'$ and $D'F'$, respectively.

To Prove $$\frac{AB}{A'B'} = \frac{EF}{E'F'}.$$

$$\left(\text{By § 253, 2, } \frac{AB}{A'B'} = \frac{CD}{C'D'}.\right)$$

529. Cor. I. *Any two homologous faces of two similar polyedrons are to each other as the squares of any two homologous edges.*

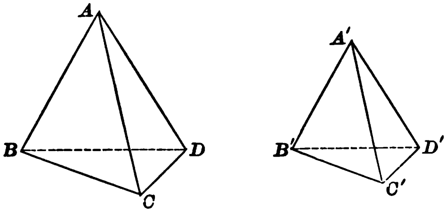
(To prove $\dfrac{\text{area } ABCD}{\text{area } A'B'C'D'} = \dfrac{\overline{EF}^2}{\overline{E'F'}^2}$. See § 322.)

530. Cor. II. *The entire surfaces of two similar polyedrons are to each other as the squares of any two homologous edges.*

(To prove $\dfrac{\text{area } ABCD + \text{area } CDEF \text{ etc.}}{\text{area } A'B'C'D' + \text{area } C'D'E'F' \text{ etc.}} = \dfrac{\overline{EF}^2}{\overline{E'F'}^2}$.)

Prop. XXV. Theorem.

531. *Two tetraedrons are similar when the faces including a triedral angle of one are similar, respectively, to the faces including a triedral angle of the other, and similarly placed.*



Given, in tetraedrons $ABCD$ and $A'B'C'D'$, face ABC similar to $A'B'C'$, ACD to $A'C'D'$, and ADB to $A'D'B'$.

To Prove $ABCD$ and $A'B'C'D'$ similar.

Proof. From the given similar faces, we have

$$\frac{BC}{B'C'} = \frac{AC}{A'C'} = \frac{CD}{C'D'} = \frac{AD}{A'D'} = \frac{BD}{B'D'}. \qquad (?)$$

Hence, faces BCD and $B'C'D'$ are similar. (§ 259)

Again, $\angle$s BAC, CAD, and DAB are equal, respectively, to $\angle$s $B'A'C'$, $C'A'D'$, and $D'A'B'$. (?)

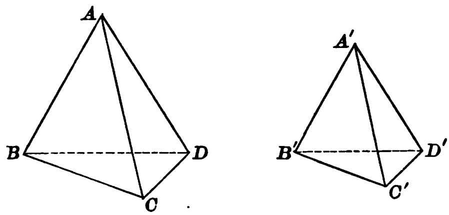
Then, triedral $\angle$s $A\text{–}BCD$ and $A'\text{–}B'C'D'$ are equal. (§ 460, 1)

Similarly, any two homologous triedral $\angle$s are equal.

Therefore, $ABCD$ and $A'B'C'D'$ are similar (§ 527).

Prop. XXVI. Theorem.

532. *Two tetraedrons are similar when a diedral angle of one is equal to a diedral angle of the other, and the faces including the equal diedral angles similar each to each, and similarly placed.*



Given, in tetraedrons $ABCD$ and $A'B'C'D'$, diedral $\angle AB$ equal to diedral $\angle A'B'$; and faces ABC and ABD similar to faces $A'B'C'$ and $A'B'D'$, respectively.

To Prove $ABCD$ and $A'B'C'D'$ similar.

Proof. Apply tetraedron $A'B'C'D'$ to $ABCD$ so that diedral $\angle A'B'$ shall coincide with its equal diedral $\angle AB$, point A' falling at A.

Then since $\angle B'A'C' = \angle BAC$ and $\angle B'A'D' = \angle BAD$, edge $A'C'$ will coincide with edge AC, and $A'D'$ with AD.

$$\therefore \angle C'A'D' = \angle CAD.$$

Again, from the given similar faces,

$$\frac{A'C'}{AC} = \frac{A'B'}{AB} = \frac{A'D'}{AD}. \qquad (?)$$

Hence, $\triangle C'A'D'$ is similar to $\triangle CAD$. (§ 261)

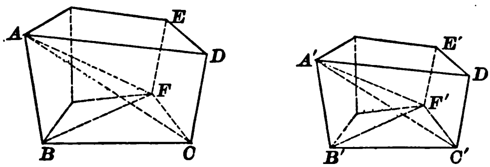
Then, the faces including triedral $\angle A'\text{-}B'C'D'$ are similar respectively to the faces including triedral $\angle A\text{-}BCD$, and similarly placed.

Therefore, $ABCD$ and $A'B'C'D'$ are similar. (§ 531)

Ex. 75. If a tetraedron be cut by a plane parallel to one of its faces, the tetraedron cut off is similar to the given tetraedron.

Prop. XXVII. Theorem.

533. *Two similar polyedrons may be decomposed into the same number of tetraedrons, similar each to each, and similarly placed.*



Given AF and $A'F'$ similar polyedrons, vertices A and A' being homologous.

To Prove that they may be decomposed into the same number of tetraedrons, similar each to each, and similarly placed.

Proof. Divide all the faces of AF, except the ones having A as a vertex, into △; and draw lines from A to their vertices.

In like manner, divide all the faces of $A'F'$, except the ones having A' as a vertex, into △ similar to those in AF, and similarly placed. (§ 267)

Draw lines from A' to their vertices.

Then, the given polyedrons are decomposed into the same number of tetraedrons, similarly placed.

Let $ABCF$ and $A'B'C'F'$ be homologous tetraedrons.

△ ABC and BCF are similar, respectively, to △ $A'B'C'$ and $B'C'F'$. (§ 267)

And since the given polyedrons are similar, the homologous diedral ∠ BC and $B'C'$ are equal.

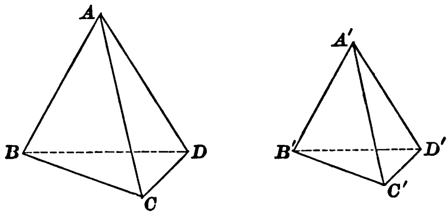
Therefore, $ABCF$ and $A'B'C'F'$ are similar. (§ 532)

In like manner, we may prove any two homologous tetraedrons similar.

Hence, the given polyedrons are decomposed into the same number of tetraedrons, similar each to each, and similarly placed.

PROP. XXVIII. THEOREM.

534. *Two similar tetraedrons are to each other as the cubes of their homologous edges.*



Given V and V' the volumes of similar tetraedrons $ABCD$ and $A'B'C'D'$, vertices A and A' being homologous.

To Prove $$\frac{V}{V'}=\frac{\overline{AB}^3}{\overline{A'B'}^3}.$$

Proof. Since the triedral $\angle$s at A and A' are equal,

$$\frac{V}{V'}=\frac{AB\times AC\times AD}{A'B'\times A'C'\times A'D'} \qquad (\S\ 523)$$

$$=\frac{AB}{A'B'}\times\frac{AC}{A'C'}\times\frac{AD}{A'D'}.$$

But, $$\frac{AC}{A'C'}=\frac{AB}{A'B'}, \text{ and } \frac{AD}{A'D'}=\frac{AB}{A'B'}. \qquad (\S\ 528)$$

$$\therefore \frac{V}{V'}=\frac{AB}{A'B'}\times\frac{AB}{A'B'}\times\frac{AB}{A'B'}=\frac{\overline{AB}^3}{\overline{A'B'}^3}.$$

535. Cor. *Any two similar polyedrons are to each other as the cubes of their homologous edges.*

For any two similar polyedrons may be decomposed into the same number of tetraedrons, similar each to each (§ 533).

Any two homologous tetraedrons are to each other as the cubes of their homologous edges. (§ 534)

Then, any two homologous tetraedrons are to each other as the cubes of any two homologous edges of the polyedrons. (§ 528)

REGULAR POLYEDRONS.

536. Def. A *regular polyedron* is a polyedron whose faces are equal regular polygons, and whose polyedral angles are all equal.

PROP. XXIX. THEOREM.

537. *Not more than five regular convex polyedrons are possible.*

A convex polyedral ∠ must have at least three faces, and the sum of its face ∠s must be $< 360°$ (§ 458).

1. *With equilateral triangles.*

Since the ∠ of an equilateral △ is 60°, we may form a convex polyedral ∠ by combining either 3, 4, or 5 equilateral △s.

Not more than 5 equilateral △s can be combined to form a convex polyedral ∠. (§ 458)

Hence, not more than three regular convex polyedrons can be bounded by equilateral △s.

2. *With squares.*

Since the ∠ of a square is 90°, we may form a convex polyedral ∠ by combining 3 squares.

Not more than 3 squares can be combined to form a convex polyedral ∠. (?)

Hence, not more than one regular convex polyedron can be bounded by squares.

3. *With regular pentagons.*

Since the ∠ of a regular pentagon is 108°, we may form a convex polyedral ∠ by combining 3 regular pentagons.

Not more than 3 regular pentagons can be combined to form a convex polyedral ∠. (?)

Hence, not more than one regular convex polyedron can be bounded by regular pentagons.

Since the ∠ of a regular hexagon is 120°, no convex polyedral ∠ can be formed by combining regular hexagons. (?)

Hence, no regular convex polyedron can be bounded by regular hexagons.

In like manner, no regular convex polyedron can be bounded by regular polygons of more than six sides.

Therefore, not more than five regular convex polyedrons are possible.

Prop. XXX. Theorem.

538. *With a given edge, to construct a regular polyedron.*

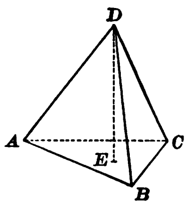
We will now prove, by actual construction, that five regular convex polyedrons are possible:

1. The regular tetraedron, bounded by 4 equilateral $\triangle$.
2. The regular hexaedron, or cube, bounded by 6 squares.
3. The regular octaedron, bounded by 8 equilateral $\triangle$.
4. The regular dodecaedron, bounded by 12 regular pentagons.
5. The regular icosaedron, bounded by 20 equilateral $\triangle$.

1. *To construct a regular tetraedron.*

Given line AB.

Required to construct with AB as an edge a regular tetraedron.



Construction. Construct the equilateral $\triangle ABC$.

At its centre E, draw line $ED \perp ABC$; and take point D so that $AD = AB$.

Draw lines AD, BD, and CD.

Then, solid $ABCD$ is a regular tetraedron.

Proof. Since A, B, and C are equally distant from E,

$$AD = BD = CD. \qquad (\S\ 406,\ \text{I})$$

Hence, the six edges of the tetraedron are all equal.

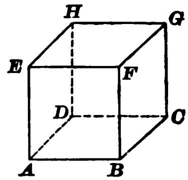
Then, the faces are equal equilateral $\triangle$. (§ 69)

And since the $\angle$ of the faces are all equal, the triedral $\angle$ whose vertices are A, B, C, and D are all equal. (§ 460, 1)

Therefore, solid $ABCD$ is a regular tetraedron. (§ 536)

2. *To construct a regular hexaedron, or cube.*

Given line AB.



Required to construct with AB as an edge a cube.

Construction. Construct square $ABCD$; and draw lines AE, BF, CG, and DH, each equal to AB, and $\perp$ $ABCD$.

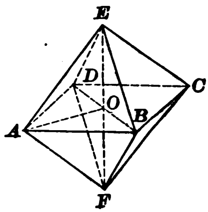
Draw lines EF, FG, GH, and HE; then, solid AG is a cube.

Proof. By cons., its faces are equal squares.

Hence, its triedral $\angle$s are all equal. (§ 460, 1)

3. *To construct a regular octaedron.*

Given line AB.



Required to construct with AB as an edge a regular octaedron.

Construction. Construct the square $ABCD$; through its centre O draw line $EF \perp ABCD$, making $OE = OF = OA$.

Draw lines EA, EB, EC, ED, FA, FB, FC, and FD; then solid $AEFC$ is a regular octaedron.

Proof. Draw lines OA, OB, and OD.

Then in rt. $\triangle$s AOB, AOE, and AOF, by cons.,

$$OA = OB = OE = OF.$$

$$\therefore \triangle AOB = \triangle AOE = \triangle AOF. \quad (?)$$

$$\therefore AB = AE = AF. \quad (?)$$

Then, the eight edges terminating at E and F are all equal. (§ 406, I)

Thus, the twelve edges of the octaedron are all equal, and the faces are equal equilateral $\triangle$s. (?)

Again, by cons., the diagonals of quadrilateral $BEDF$ are equal, and bisect each other at rt. $\angle$s.

Hence, $BEDF$ is a square equal to $ABCD$, and OA is $\perp$ to its plane. (§ 400)

Then, pyramids A–$BEDF$ and E–$ABCD$ are equal; and hence polyedral $\angle$s A–$BEDF$ and E–$ABCD$ are equal.

In like manner, any two polyedral $\angle$s are equal.

Therefore, solid $AEFC$ is a regular octaedron.

4. *To construct a regular dodecaedron.*

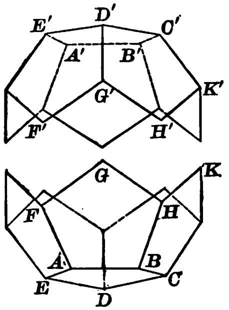


Fig. 1.

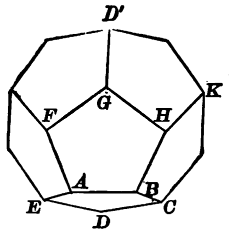


Fig. 2.

Given line AB.

Required to construct with AB as an edge a regular dodecaedron.

Construction. Construct regular pentagon $ABCDE$ (Fig. 1); and to it join five equal regular pentagons, so inclined as to form equal triedral $\angle$s at A, B, C, D, and E. (§ 460, 1)

Then there is formed a convex surface AK composed of six regular pentagons, as shown in lower part of Fig. 1.

Construct a second surface $A'K'$ equal to AK, as shown in upper part of Fig. 1.

Surfaces AK and $A'K'$ may be combined as shown in Fig. 2, so as to form at F a triedral $\angle$ equal to that at A, having for its faces the regular pentagons about vertices F and F' in Fig. 1. (§ 460, 1)

Then, solid AK is a regular dodecaedron.

Proof. Since G' falls at G, and diedral $\angle FG$ and face $\angle$s FGH and FGD' (Fig. 2) are equal respectively to the diedral $\angle$ and face $\angle$s of triedral $\angle F$, the faces about vertex G will form a triedral $\angle$ equal to that at F.

In this way, it may be proved that at each of the vertices H, K, etc., there is formed a triedral $\angle$ equal to that at F.

Therefore, solid AK is a regular dodecaedron.

5. *To construct a regular icosaedron.*

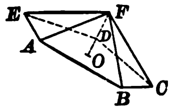


Fig. 1.

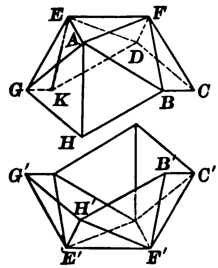


Fig. 2.

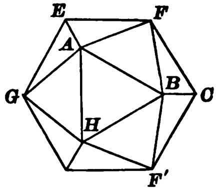


Fig. 3.

Given line AB.

Required to construct with AB as an edge a regular icosaedron.

Construction. Construct regular pentagon $ABCDE$ (Fig. 1); at its centre O draw line $OF \perp ABCDE$, making $AF = AB$, and draw lines AF, BF, CF, DF, and EF.

Then, $F\text{–}ABCDE$ is a polyedral $\angle$ composed of five equal equilateral $\triangle$. (§§ 406, I, 69)

Then construct two other polyedral $\angle$s, $A\text{–}BFEGH$ and $E\text{–}AFDKG$, each equal to $F\text{–}ABCDE$; and place them as shown in upper part of Fig. 2, so that faces ABF and AEF of $A\text{–}BFEGH$, and faces AEF and DEF of $E\text{–}AFDKG$, shall coincide with the corresponding faces of $F\text{–}ABCDE$.

Then there is formed a convex surface GC, composed of ten equilateral $\triangle$.

Construct a second surface $G'C'$ equal to GC, as shown in lower part of Fig. 2.

Surfaces GC and $G'C'$ may be combined as shown in Fig. 3, so that edges GH and HB shall coincide with edges $G'H'$ and $H'B'$, respectively.

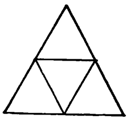
Then, solid GC is a regular icosaedron.

Proof. Since diedral $\angle$s AH, $E'H'$, and $F'H'$ are equal to the diedral $\angle$s of polyedral $\angle F$, the faces about vertices H and H' form a polyedral $\angle$ at H equal to that at F.

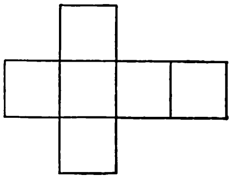
Then, since diedral $\angle$s *FB*, *AB*, *HB*, and *F'B* (Fig. 3) are equal to the diedral $\angle$s of polyedral $\angle F$, the faces about vertex *B* form a polyedral $\angle$ equal to that at *F*; and it may be shown that at each of the vertices *C*, *D*, etc., there is formed a polyedral $\angle$ equal to that at *F*.

Therefore, solid *GC* is a regular icosaedron.

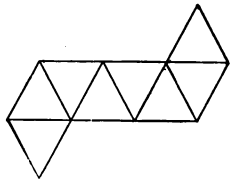
539. Sch. To construct the regular polyedrons, draw the following figures on cardboard; cut them out entire, and on the interior lines cut the cardboard half through; the edges may then be brought together to form the respective solids.



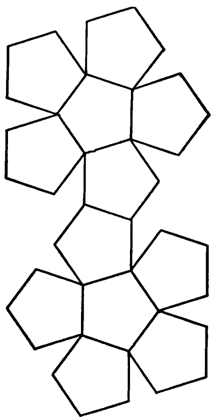
TETRAEDRON.



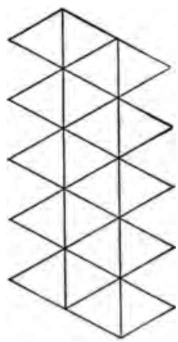
HEXAEDRON.



OCTAEDRON.



DODECAEDRON.



ICOSAEDRON.

EXERCISES.

76. The volume of a pyramid whose altitude is 7 in. is 686 cu. in. Find the volume of a similar pyramid whose altitude is 12 in.

77. If the volume of a prism whose altitude is 9 ft. is 171 cu. ft., find the altitude of a similar prism whose volume is $50\frac{2}{3}$ cu. ft.

(Represent the altitude by x.)

78. Two bins of similar form contain, respectively, 375 and 648 bushels of wheat. If the first bin is 3 ft. 9 in. long, what is the length of the second?

79. A pyramid whose altitude is 10 in., weighs 24 lb. At what distance from its vertex must it be cut by a plane parallel to its base so that the frustum cut off may weigh 12 lb.?

80. An edge of a polyedron is 56, and the homologous edge of a similar polyedron is 21. The area of the entire surface of the second polyedron is 135, and its volume is 162. Find the area of the entire surface, and the volume, of the first polyedron.

81. The area of the entire surface of a tetraedron is 147, and its volume is 686. If the area of the entire surface of a similar tetraedron is 48, what is its volume?

(Let x and y denote the homologous edges of the tetraedrons.)

82. The area of the entire surface of a tetraedron is 75, and its volume is 500. If the volume of a similar tetraedron is 32, what is the area of its entire surface?

83. The homologous edges of three similar tetraedrons are 3, 4, and 5, respectively. Find the homologous edge of a similar tetraedron equivalent to their sum.

(Represent the edge by x.)

84. State and prove the converse of Prop. XXVII.

85. The volume of a regular tetraedron is equal to the cube of its edge multiplied by $\frac{1}{12}\sqrt{2}$.

86. The volume of a regular tetraedron is $18\sqrt{2}$. Find the area of its entire surface. (Ex. 85.)

(Represent the edge by x.)

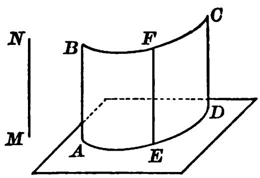
87. The volume of a regular octaedron is equal to the cube of its edge multiplied by $\frac{1}{3}\sqrt{2}$.

Book VIII.

THE CYLINDER, CONE, AND SPHERE.

DEFINITIONS.

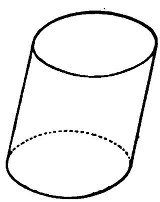
540. A *cylindrical surface* is a surface generated by a moving straight line, which constantly intersects a given plane curve, and in all its positions is parallel to a given straight line, not in the plane of the curve.



Thus, if line *AB* moves so as to constantly intersect plane curve *AD*, and is constantly parallel to line *MN*, not in the plane of the curve, it generates a cylindrical surface.

The moving line is called the *generatrix*, and the curve the *directrix*.

Any position of the generatrix, as *EF*, is called an *element* of the surface.



A *cylinder* is a solid bounded by a cylindrical surface, and two parallel planes.

The parallel planes are called the *bases* of the cylinder, and the cylindrical surface the *lateral surface*.

The *altitude* of a cylinder is the perpendicular distance between the planes of its bases.

A *right cylinder* is a cylinder the elements of whose lateral surface are perpendicular to its bases.

A *circular cylinder* is a cylinder whose base is a circle.

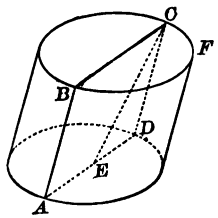
A plane is said to be *tangent* to a cylinder when it contains one, and only one, element of the lateral surface.

541. It follows from the definition of a cylinder (§ 540) that

The elements of the lateral surface of a cylinder are equal and parallel. (§ 415)

PROP. I. THEOREM.

542. *A section of a cylinder made by a plane passing through an element of the lateral surface is a parallelogram.*



Given *ABCD* a section of cylinder *AF*, made by a plane passing through *AB*, an element of the lateral surface.

To Prove section *ABCD* a ▱.

Note. It should be observed that, with the above hypothesis, *CD* simply represents the intersection of plane *AC* with the cylindrical surface, and may be a *curved* line; it must be proved that it is a str. line ∥ *AB*.

Proof. *AD* and *BC* are str. lines, and ∥. (§§ 396, 414)

Now draw str. line *CE* in plane *AC* ∥ *AB*; then, *CE* is an element of the cylindrical surface. (§§ 541, 53)

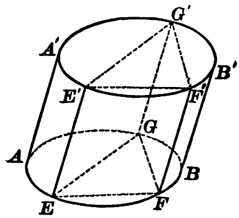
Then since *CE* lies in plane *AC*, and also in the cylindrical surface, it must be the intersection of the plane with the cylindrical surface.

Then, *CD* is a str. line ∥ *AB*, and *ABCD* is a ▱.

543. Cor. *A section of a right cylinder made by a plane perpendicular to its base is a rectangle.*

PROP. II. THEOREM.

544. *The bases of a cylinder are equal.*



Given cylinder AB'.

To Prove base $A'B'$ = base AB.

Proof. Let E', F', and G' be any three points in the perimeter of base $A'B'$, and draw EE', FF', and GG' elements of the lateral surface.

Draw lines EF, FG, GE, $E'F'$, $F'G'$, and $G'E'$.

Now, EE' and FF' are equal and ‖. (§ 541)

Then, $EE'F'F$ is a ▱. (?)

$\therefore E'F' = EF$. (?)

Similarly, $E'G' = EG$ and $F'G' = FG$.

$\therefore \triangle E'F'G' = \triangle EFG$. (?)

Then, base $A'B'$ may be superposed upon base AB so that points E', F', and G' shall fall at E, F, and G, respectively.

But E' is any point in the perimeter of $A'B'$.

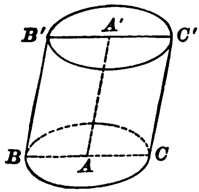
Then, *every* point in the perimeter of $A'B'$ will fall somewhere in the perimeter of AB, and base $A'B'$ = base AB.

545. Cor. I. *The sections of a circular cylinder made by planes parallel to its bases are equal circles.*

For each may be regarded as the upper base of a cylinder whose lower base is a ⊙.

546. Def. The *axis* of a circular cylinder is a straight line drawn between the centres of its bases.

547. Cor. II. *The axis of a circular cylinder is parallel to the elements of its lateral surface.*



Given AA' the axis, and BB' an element of the lateral surface, of circular cylinder BC'.

To Prove $AA' \parallel BB'$.

Proof. Let $BB'C'C$ be a section made by a plane passing through BB' and A; then $BB'C'C$ is a ▱. (§ 542)

$\therefore B'C' = BC$. (?)

Then since BC is a diameter of ⊙ BC, and Ⓢ BC and $B'C'$ are equal, $B'C'$ is a diameter of ⊙ $B'C'$, and passes through A'.

Hence, AB and $A'B'$ are equal and ∥. (?)

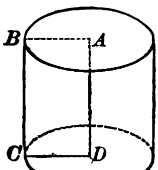
Then, $ABB'A'$ is a ▱. (?)

$\therefore AA' \parallel BB'$.

548. Cor. III. *The axis of a circular cylinder passes through the centres of all sections parallel to the bases.*

PROP. III. THEOREM.

549. *A right circular cylinder may be generated by the revolution of a rectangle about one of its sides as an axis.*



Given rect. $ABCD$.

To Prove the solid generated by the revolution of $ABCD$ about AD as an axis a rt. circular cylinder.

Proof. All positions of BC are ∥ AD.

Again, AB and CD generate Ⓢ ⊥ AD. (§ 402)

Then, these Ⓢ are ∥, and ⊥ BC. (§§ 421, 419)

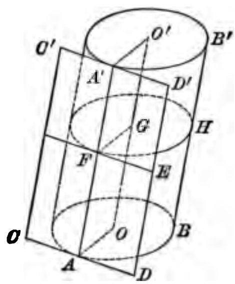
Whence, $ABCD$ generates a rt. circular cylinder.

550. Defs. From the property proved in § 549, a right circular cylinder is called a *cylinder of revolution.*

Similar cylinders of revolution are cylinders generated by the revolution of similar rectangles about homologous sides as axes.

PROP. IV. THEOREM.

551. *A plane drawn through an element of the lateral surface of a circular cylinder and a tangent to the base at its extremity, is tangent to the cylinder.*



Given AA' an element of the lateral surface of circular cylinder AB', line CD tangent to base AB at A, and plane CD' drawn through AA' and CD.

To Prove CD' tangent to the cylinder.

Proof. Let E be any point in plane CD', not in AA', and draw through E a plane $\parallel$ to the bases, intersecting CD' in line EF, and the cylinder in $\odot$ FH. (§ 545)

Draw axis OO'; then OO' is $\parallel$ AA'. (§ 547)

Let the plane of OO' and AA' intersect the planes of AB and FH in radii OA and GF, respectively. (§ 548)

Then, $GF \parallel OA$ and $FE \parallel AD$. (§ 414)

$\therefore \angle GFE = \angle OAD$. (§ 426)

But $\angle OAD$ is a rt. $\angle$. (§ 170)

Then, FE is $\perp$ GF, and tangent to $\odot$ FH. (§ 169)

Whence, point E lies without the cylinder.

Then, all portions of CD', not in AA', lie without the cylinder, and CD' is tangent to the cylinder.

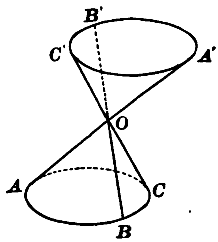
552. Cor. *A plane tangent to a circular cylinder intersects the planes of the bases in lines which are tangent to the bases.*

Ex. 1. The sections of a cylinder made by two parallel planes which cut all the elements of its lateral surface are equal.

THE CONE.

DEFINITIONS.

553. A *conical surface* is a surface generated by a moving straight line, which constantly intersects a given plane curve, and passes through a given point not in the plane of the curve.



Thus, if line OA moves so as to constantly intersect plane curve ABC, and constantly passes through point O, not in the plane of the curve, it generates a conical surface.

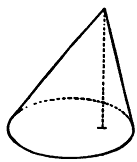
The moving line is called the *generatrix*, and the curve the *directrix*.

The given point is called the *vertex*, and any position of the generatrix, as OB, is called an *element* of the surface.

If the generatrix be supposed indefinite in length, it will generate two conical surfaces of indefinite extent, $O\text{–}A'B'C'$ and $O\text{–}ABC$.

These are called the *upper* and *lower nappes*, respectively.

A *cone* is a solid bounded by a conical surface, and a plane cutting all its elements.



The plane is called the *base* of the cone, and the conical surface the *lateral surface*.

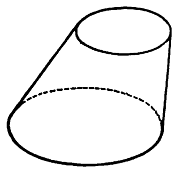
The *altitude* of a cone is the perpendicular distance from the vertex to the plane of the base.

A *circular cone* is a cone whose base is a circle.

The *axis* of a circular cone is a straight line drawn from the vertex to the centre of the base.

A *right circular cone* is a circular cone whose axis is perpendicular to its base.

A *frustum of a cone* is a portion of a cone included between the base and a plane parallel to the base.



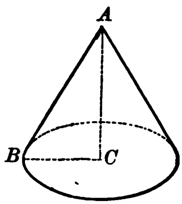
The base of the cone is called the *lower base*, and the section made by the plane the *upper base*, of the frustum.

The *altitude* is the perpendicular distance between the planes of the bases.

A plane is said to be *tangent* to a cone, or frustum of a cone, when it contains one, and only one, element of the lateral surface.

PROP. V. THEOREM.

554. *A right circular cone may be generated by the revolution of a right triangle about one of its legs as an axis.*



Given C the rt. $\angle$ of rt. $\triangle\, ABC$.

To Prove the solid generated by the revolution of ABC about AC as an axis a right circular cone.

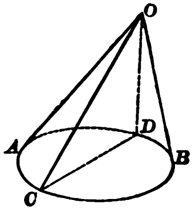
(The proof is left to the pupil.)

555. Defs. From the above property, a right circular cone is called a *cone of revolution*.

Similar cones of revolution are cones generated by the revolution of similar right triangles about homologous legs as axes.

PROP. VI. THEOREM.

556. *A section of a cone made by a plane passing through the vertex is a triangle.*



Given *OCD* a section of cone *OAB* made by a plane passing through vertex *O*.

To Prove section *OCD* a △.

Proof. We have *CD* a str. line. (§ 396)

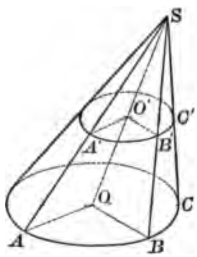
Now draw str. lines in plane *OCD* from *O* to *C* and *D*; these str. lines are elements of the conical surface. (§ 550)

Then, since these str. lines lie in plane *OCD*, and also in the conical surface, they must be the intersections of the plane with the conical surface.

Then, *OC* and *OD* are str. lines, and *OCD* is a △.

PROP. VII. THEOREM.

557. *A section of a circular cone made by a plane parallel to the base is a circle.*



Given *A'B'C'* a section of circular cone *S–ABC*, made by a plane ∥ to the base.

To Prove *A'B'C'* a ⊙.

Proof. Draw axis OS, intersecting plane $A'B'C'$ at O'.

Let A' and B' be any two points in perimeter $A'B'C'$.

Let the planes determined by these points and OS intersect the base in radii OA and OB, the section in lines $O'A'$ and $O'B'$, and the lateral surface in lines $SA'A$ and $SB'B$, respectively.

Then, $SA'A$ and $SB'B$ are str. lines. (§ 556)

Now, $O'A' \parallel OA$ and $O'B' \parallel OB$. (§ 414)

Then, $\triangle SO'A'$ and $SO'B'$ are similar to $\triangle SOA$ and SOB, respectively. (§ 257)

$$\therefore \frac{O'A'}{OA} = \frac{SO'}{SO} \text{ and } \frac{O'B'}{OB} = \frac{SO'}{SO}. \qquad (?)$$

$$\therefore \frac{O'A'}{OA} = \frac{O'B'}{OB}. \qquad (?)$$

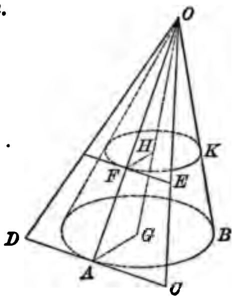
But, $OA = OB$. (§ 143)

Then, $O'A' = O'B'$; and as A' and B' are *any* two points in perimeter $A'B'C'$, section $A'B'C'$ is a $\odot$.

558. Cor. *The axis of a circular cone passes through the centre of every section parallel to the base.*

Prop. VIII. Theorem.

559. *A plane drawn through an element of the lateral surface of a circular cone and a tangent to the base at its extremity, is tangent to the cone.*



Given OA an element of the lateral surface of circular cone OAB, line CD tangent to base AB at A, and plane OCD drawn through OA and CD.

To Prove OCD tangent to the cone.

(Prove that E lies without the cone.)

560. Cor. *A plane tangent to a circular cone intersects the plane of the base in a line tangent to the base.*

THE SPHERE.

DEFINITIONS.

561. A *sphere* is a solid bounded by a surface, all points of which are equally distant from a point within called the *centre.*

A *radius* of a sphere is a straight line drawn from the centre to the surface.

A *diameter* is a straight line drawn through the centre, having its extremities in the surface.

562. It follows from the definition of § 561 that *all radii of a sphere are equal.*

Also, all its diameters are equal, since each is the sum of two radii.

563. *Two spheres are equal when their radii are equal.*

For they can evidently be applied one to the other so that their surfaces shall coincide throughout.

Conversely, *the radii of equal spheres are equal.*

564. A line (or a plane) is said to be *tangent to a sphere* when it has one, and only one, point in common with the surface; the common point is called the *point of contact.*

A polyedron is said to be *inscribed in a sphere* when all its vertices lie in the surface of the sphere; in this case the sphere is said to be *circumscribed about the polyedron.*

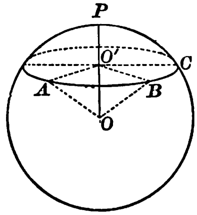
A polyedron is said to be *circumscribed about a sphere* when all its faces are tangent to the sphere; in this case the sphere is said to be *inscribed in the polyedron.*

565. *A sphere may be generated by the revolution of a semicircle about its diameter as an axis.*

For all points of such a surface are equally distant from the centre of the ⊙. (?)

PROP. IX. THEOREM.

566. *A section of a sphere made by a plane is a circle.*



Given ABC a section of sphere APC made by a plane.

To Prove ABC a ⊙.

Proof. Let O be the centre of the sphere, and draw line $OO' \perp$ to plane ABC.

Let A and B be any two points in perimeter ABC, and draw lines OA, OB, $O'A$, and $O'B$.

Now, $OA = OB$. (?)

$\therefore O'A = O'B$. (§ 407, I)

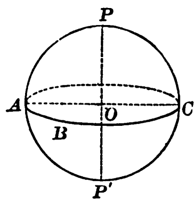
But A and B are *any* two points in perimeter ABC.

Therefore, ABC is a ⊙.

567. Defs. A *great circle* of a sphere is a section made by a plane passing through the centre; as ABC.

A *small circle* is a section made by a plane which does not pass through the centre.

The *diameter* perpendicular to a circle of a sphere is called the *axis* of the circle, and its extremities are called the *poles*.



568. Cor. I. *The axis of a circle of a sphere passes through the centre of the circle.*

569. Cor. II. *All great circles of a sphere are equal.*

For their radii are radii of the sphere.

570. Cor. III. *Every great circle bisects the sphere and its surface.*

For if the portions of the sphere formed by the plane of the great ⊙ be separated, and placed so that their plane surfaces coincide, the spherical surfaces falling on the same side of this plane, the two spherical surfaces will coincide throughout; for all points of either surface are equally distant from the centre.

571. Cor. IV. *Any two great circles bisect each other.*

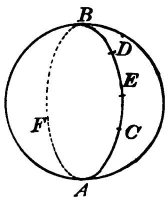
For the intersection of their planes is a diameter of the sphere, and therefore a diameter of each ⊙. (§ 152)

572. Cor. V. *Between any two points on the surface of a sphere, not the extremities of a diameter, an arc of a great circle, less than a semi-circumference, can be drawn, and but one.*

For the two points, with the centre of the sphere, determine a plane which intersects the surface of the sphere in the required arc.

Note. If the points *are* the extremities of a diameter, an indefinitely great number of arcs of great Ⓢ can be drawn between them; for an indefinitely great number of planes can be drawn through the diameter.

573. Def. The *distance* between two points on the surface of a sphere, not at the extremities of a diameter, is the arc of a great circle, less than a semi-circumference, drawn between them.



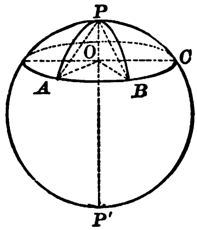
Thus, the distance between points C and D is arc CED, and not arc $CAFBD$.

574. Cor. VI. *An arc of a circle may be drawn through any three points on the surface of a sphere.*

For the three points determine a plane which intersects the surface of the sphere in the required arc.

PROP. X. THEOREM.

575. *All points in the circumference of a circle of a sphere are equally distant from each of its poles.*



Given P and P' the poles of ⊙ ABC of sphere APC.

To Prove all points in circumference ABC equally distant (§ 573) from P, and also from P'.

Proof. Let A and B be any two points in circumference ABC, and draw arcs of great ⓢ PA and PB.

Draw axis PP', intersecting plane ABC at O.

Draw lines OA and OB, and chords PA and PB.

Now O is the centre of ⊙ ABC. (§ 568)

$\therefore OA = OB$. (?)

$\therefore$ chord $PA =$ chord PB. (§ 406, I)

$\therefore$ arc $PA =$ arc PB. (§ 157)

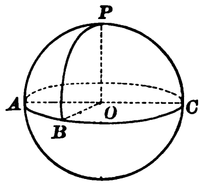
But A and B are *any* two points in circumference ABC.

Therefore, *all* points in circumference ABC are equally distant from P.

In like manner, all points in circumference ABC are equally distant from P'.

576. Def. The *polar distance* of a circle of a sphere is the distance (§ 573) from the nearer of its poles, or from either pole if they are equally near, to the circumference.

Thus, in figure of Prop. X, the polar distance of ⊙ ABC is arc PA.



577. Cor. *All points in the circumference of a great circle of a sphere are at a quadrant's distance from either pole.*

Given P a pole of great ⊙ ABC of sphere APC, B any point in circumference ABC, and PB an arc of a great ⊙.

To Prove arc PB a quadrant (§ 146).

Proof. Let O be the centre of the sphere, and draw radii OB and OP.

Then, $\angle POB$ is a rt. $\angle$. (§ 398)

Whence, arc PB is a quadrant. (§ 191)

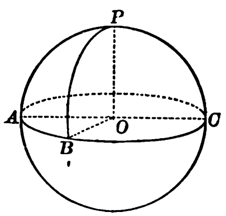
The above proof holds for either pole of the great ⊙.

Note. An arc of a circle may be drawn on the surface of a sphere by placing one foot of the compasses at the nearer pole of the circle, the distance between the feet being equal to the chord of the polar distance.

PROP. XI. THEOREM.

578. *If a point on the surface of a sphere lies at a quadrant's distance from each of two points in the arc of a great circle, it is a pole of that arc.*

Note. The term *quadrant*, in Spherical Geometry, usually signifies a quadrant of a great circle.



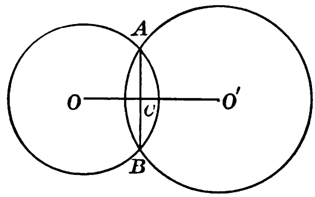
Given point P on surface of sphere APC, AB an arc of great ⊙ ABC, and PA and PB quadrants.

To Prove P a pole of arc AB.

(PO is ⊥ to OA and OB; then use § 400.)

Prop. XII. Theorem.

579. *The intersection of two spheres is a circle, whose centre is in the straight line joining the centres of the spheres, and whose plane is perpendicular to that line.*



Given two intersecting spheres.

To Prove their intersection a ⊙, whose centre is in the line joining the centres of the spheres, and whose plane is ⊥ to this line.

Proof. Let O and O' be the centres of two ⓢ, whose common chord is AB; draw line OO', intersecting AB at C.

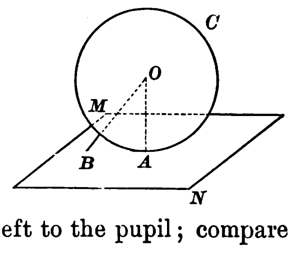
Then, OO' bisects AB at rt. ∡s. (§ 178)

If we revolve the entire figure about OO' as an axis, the ⓢ will generate spheres whose centres are O and O'. (§ 565)

And AC will generate a ⊙ ⊥ OO', whose centre is C, which is the intersection of the two spheres. (§ 402)

Prop. XIII. Theorem.

580. *A plane perpendicular to a radius of a sphere at its extremity is tangent to the sphere.*



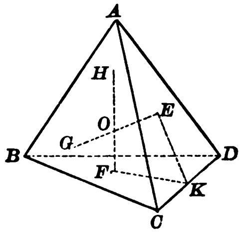
(The proof is left to the pupil; compare § 169.)

581. Cor. (Converse of Prop. XIII.) *A plane tangent to a sphere is perpendicular to the radius drawn to the point of contact.* (Fig. of Prop. XIII.)

(The proof is left to the pupil; compare § 170.)

Prop. XIV. Theorem.

582. *Through four points, not in the same plane, a spherical surface can be made to pass, and but one.*



Given *A, B, C,* and *D* points not in the same plane.

To Prove that a spherical surface can be passed through *A, B, C,* and *D,* and but one.

Proof. Pass planes through *A, B, C,* and *D,* forming tetraedron *ABCD,* and let *K* be the middle point of *CD.*

Draw lines *KE* and *KF* in faces *ACD* and *BCD,* respectively, $\perp CD$; and let *E* and *F* be the centres of the circumscribed Ⓢ of ⚠ *ACD* and *BCD,* respectively. (§ 222)

Then plane *EKF* is $\perp$ *CD.* (§ 400)

Draw line $EG \perp ACD$, and line $FH \perp BCD$; then *EG* and *FH* lie in plane *EKF.* (§ 439)

Then *EG* and *FH* must meet at some point *O,* unless they are ∥; this cannot be unless *ACD* and *BCD* are in the same plane, which is contrary to the hyp. (§ 418)

Now *O,* being in *EG,* is equally distant from *A, C,* and *D*; and being in *FH,* is equally distant from *B, C,* and *D.* (§ 406, I)

Then *O* is equally distant from *A, B, C,* and *D*; and a spherical surface described with *O* as a centre, and *OA* as a radius, will pass through *A, B, C,* and *D.*

Now the centre of any spherical surface passing through A, B, C, and D must be in each of the ⊥s EG and FH.

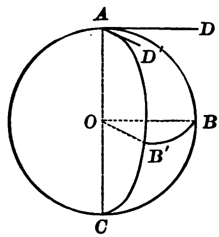
Then as EG and FH intersect in but one point, only one spherical surface can be passed through A, B, C, and D.

583. Defs. *The angle between two intersecting curves* is the angle between tangents to the curves at their point of intersection.

A *spherical angle* is the angle between two intersecting arcs of great circles.

PROP. XV. THEOREM.

584. *A spherical angle is measured by an arc of a great circle having its vertex as a pole, included between its sides produced if necessary.*



Given ABC and $AB'C$ arcs of great ⓢ on the surface of sphere AC, lines AD and AD' tangent to ABC and $A'BC$, respectively, and BB' an arc of a great ⊙ having A as a pole, included between arcs ABC and $AB'C$.

To Prove that $\angle DAD'$ is measured by arc BB'.

Proof. Let O be the centre of the sphere, and draw diameter AOC and lines OB and OB'.

Now, arcs AB and AB' are quadrants. (§ 577)

Whence, ∠s AOB and AOB' are rt. ∠s. (?)

Therefore, $OB \parallel AD$ and $OB' \parallel AD'$. (§§ 170, 54)

$\therefore \angle DAD' = \angle BOB'$. (§ 426)

But $\angle BOB'$ is measured by arc BB'. (?)

Then, $\angle DAD'$ is measured by arc BB'.

585. Cor. I. (Fig. of Prop. XV.) Plane BOB' is $\perp$ OA. (§ 400)

Then planes ABC and BOB' are $\perp$. (§ 441)

Now a tangent to arc AB at B is $\perp$ BOB'. (§ 439)

Then it is $\perp$ to a tangent to arc BB' at B. (§ 398)

Then, spherical $\angle ABB'$ is a rt. $\angle$. (§ 583)

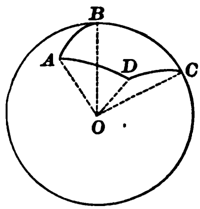
That is, *an arc of a great circle drawn from the pole of a great circle is perpendicular to its circumference.*

586. Cor. II. *The angle between two arcs of great circles is the plane angle of the diedral angle between their planes.* (§ 429)

SPHERICAL POLYGONS AND SPHERICAL PYRAMIDS.

Definitions.

587. A *spherical polygon* is a portion of the surface of a sphere bounded by three or more arcs of *great* circles; as $ABCD$.



The bounding arcs are called the *sides* of the spherical polygon, and are usually measured in *degrees*.

The *angles* of the spherical polygon are the spherical angles (§ 583) between the adjacent sides, and their vertices are called the *vertices* of the spherical polygon.

A *diagonal* of a spherical polygon is an arc of a great circle joining any two vertices which are not consecutive.

A *spherical triangle* is a spherical polygon of three sides.

A spherical triangle is called *isosceles* when it has two sides equal; *equilateral* when all its sides are equal; and *right-angled* when it has a right angle.

588. The planes of the sides of a spherical polygon form a polyedral angle, whose vertex is the centre of the sphere, and whose face angles are measured by the sides of the spherical polygon (§ 192).

Thus, in the figure of § 587, the planes of the sides of the spherical polygon form a polyedral angle, $O\text{–}ABCD$, whose face ∠s AOB, BOC, etc., are measured by arcs AB, BC, etc., respectively.

A spherical polygon is called *convex* when the polyedral angle formed by the planes of its sides is convex (§ 453).

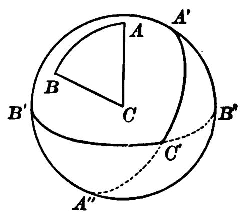
589. A *spherical pyramid* is a solid bounded by a spherical polygon and the planes of its sides; as $O\text{–}ABCD$, figure of § 587.

The centre of the sphere is called the *vertex* of the spherical pyramid, and the spherical polygon the *base*.

Two spherical pyramids are equal when their bases are equal.

For they can evidently be applied one to the other so as to coincide throughout.

590. If circumferences of great circles be drawn with the vertices of a spherical triangle as poles, they divide the surface of the sphere into eight spherical triangles.



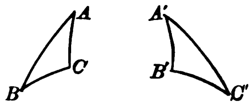
Thus, if circumference $B'C'B''$ be drawn with vertex A of spherical $\triangle ABC$ as a pole, circumference $A'C''A''$ with B as a pole, and circumference $A'B''A''B'$ with C as a pole, the surface of the sphere is divided into eight spherical ▵s; $A'B'C'$, $A'B''C'$, $A''B'C'$, and $A''B''C'$ on the hemisphere represented in the figure, the others on the opposite hemisphere.

Of these eight spherical ▵s, one is called the *polar triangle* of ABC, and is determined as follows:

Of the intersections, A' and A'', of circumferences drawn with B and C as poles, that which is nearer (§ 573) to A, *i.e.*, A', is a vertex of the polar triangle; and similarly for the other intersections.

Thus, $A'B'C'$ is the polar $\triangle$ of ABC.

591. Two spherical polygons, on the same or equal spheres, are said to be *symmetrical* when the sides and angles of one are equal, respectively, to the sides and angles of the other, if the equal parts occur *in the reverse order.*

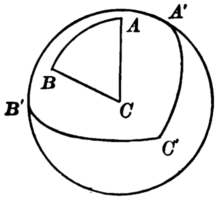


Thus, if spherical ⊿s ABC and $A'B'C'$, on the same or equal spheres, have sides AB, BC, and CA equal, respectively, to sides $A'B'$, $B'C'$, and $C'A'$, and ∠s A, B, and C to ∠s A', B', and C', and the equal parts occur in the reverse order, the ⊿s are symmetrical.

It is evident that, in general, two symmetrical spherical polygons cannot be placed so as to coincide throughout.

Prop. XVI. Theorem.

592. *If one spherical triangle is the polar triangle of another, then the second spherical triangle is the polar triangle of the first.*



Given $A'B'C'$ the polar △ of spherical △ ABC; A, B, and C being the poles of arcs $B'C'$, $C'A'$, and $A'B'$, respectively.

To Prove ABC the polar △ of spherical △ $A'B'C'$.

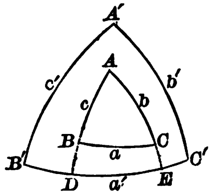
Proof. B is the pole of arc $A'C'$.
Whence, A' lies at a quadrant's distance from B. (§ 577)
Again, C is the pole of arc $A'B'$.
Whence, A' lies at a quadrant's distance from C.
Therefore, A' is the pole of arc BC. (§ 578)
Similarly, B' is the pole of arc CA, and C' of arc AB.
Then, ABC is the polar △ of $A'B'C'$.

For of the two intersections of the circumferences having B' and C', respectively, as poles, A is the nearer to A'; and similarly for the other vertices. (§ 590)

Note. Two spherical triangles, each of which is the polar triangle of the other, are called *polar triangles.*

PROP. XVII. THEOREM.

593. *In two polar triangles, each angle of one is measured by the supplement of that side of the other of which it is the pole.*



Given A, B, C, A', B', and C' the ∠s, expressed in degrees, of polar △s ABC and $A'B'C'$; A being the pole of $B'C'$, B of $C'A'$, C of $A'B'$, A' of BC, B' of CA, and C' of AB.

Let sides BC, CA, AB, $B'C'$, $C'A'$, and $A'B'$, expressed in degrees, be denoted by a, b, c, a', b', and c', respectively.

To Prove

$A = 180° - a'$, $\quad B = 180° - b'$, $\quad C = 180° - c'$,

$A' = 180° - a$, $\quad B' = 180° - b$, $\quad C' = 180° - c$.

Proof. Produce arcs AB and AC to meet arc $B'C'$ at D and E, respectively.

Since B' is the pole of arc AE, and C' of arc AD, arcs $B'E$ and $C'D$ are quadrants. (§ 577)

$$\therefore \text{ arc } B'E + \text{arc } C'D = 180°.$$

Or, $$\text{arc } DE + \text{arc } B'C' = 180°.$$

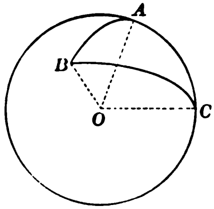
But since A is the pole of arc $B'C'$, arc DE is the measure of ∠ A. (§ 584)

$$\therefore A + a' = 180°, \text{ or } A = 180° - a'.$$

In like manner, the theorem may be proved for any ∠ of either △.

PROP. XVIII. THEOREM.

594. *Any side of a spherical triangle is less than the sum of the other two sides.*



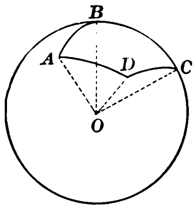
Given AB any side of spherical $\triangle ABC$.

To Prove $AB < AC + BC$.

(By § 457, $\angle AOB < \angle AOC + \angle BOC$; and these ∠s are measured by sides AB, AC, and BC, respectively.)

PROP. XIX. THEOREM.

595. *The sum of the sides of a convex spherical polygon is less than* 360°.



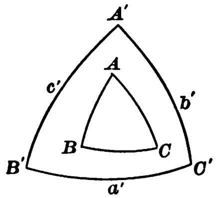
Given convex spherical polygon $ABCD$.

To Prove $AB + BC + CD + DA < 360°$.

(By § 458, sum of ∠s AOB, BOC, COD, and DOA is $< 360°$.)

PROP. XX. THEOREM.

596. *The sum of the angles of a spherical triangle is greater than two, and less than six, right angles.*



Given A, B, and C the $\angle$s, expressed in degrees, of spherical $\triangle ABC$.

To Prove $A+B+C>180°$, and $<540°$.

Proof. Let $A'B'C'$ be the polar $\triangle$ of spherical $\triangle ABC$, A being the pole of $B'C'$, B of $C'A'$, and C of $A'B'$.

Also, let sides $B'C'$, $C'A'$, and $A'B'$, expressed in degrees, be denoted by a', b', and c', respectively.

Then, $A=180°-a'$,

$B=180°-b'$,

$C=180°-c'$. (§ 593)

Adding these equations, we have

$$A+B+C=540°-(a'+b'+c'). \quad (1)$$

$$\therefore A+B+C<540°.$$

Again, $a'+b'+c'<360°$. (§ 595)

Whence, by (1), $A+B+C>180°$.

597. Cor. *A spherical triangle may have one, two, or three right angles, or one, two, or three obtuse angles.*

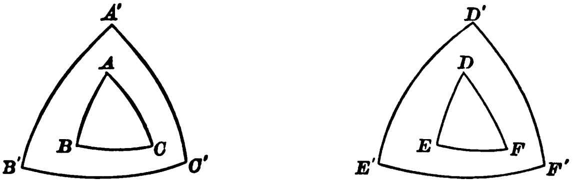
DEFINITIONS.

598. A spherical triangle having two right angles is called a *bi-rectangular triangle*, and one having three right angles a *tri-rectangular triangle.*

599. Two spherical polygons on the same sphere, or equal spheres, are said to be *mutually equilateral*, or *mutually equiangular*, when the sides or angles of one are equal, respectively, to the homologous sides or angles of the other, *whether taken in the same or in the reverse order.*

PROP. XXI. THEOREM.

600. *If two spherical triangles on the same sphere, or equal spheres, are mutually equiangular, their polar triangles are mutually equilateral.*



Given ABC and DEF mutually equiangular spherical ⊿s on the same sphere, or equal spheres, ∠s A and D being homologous; also, $A'B'C'$ the polar △ of ABC, and $D'E'F'$ of DEF, A being the pole of $B'C'$, and D of $E'F'$.

To Prove $A'B'C'$ and $D'E'F'$ mutually equilateral.

Proof. ∠s A and D are measured by the supplements of sides $B'C'$ and $E'F'$, respectively. (§ 593)

But by hyp., $\angle A = \angle D$.

$\therefore B'C' = E'F'$. (§ 31, 2)

In like manner, any two homologous sides of $A'B'C'$ and $D'E'F'$ may be proved equal.

Then, $A'B'C'$ and $D'E'F'$ are mutually equilateral.

601. Cor. (Converse of Prop. XXI.) *If two spherical triangles on the same sphere, or equal spheres, are mutually equilateral, their polar triangles are mutually equiangular.*

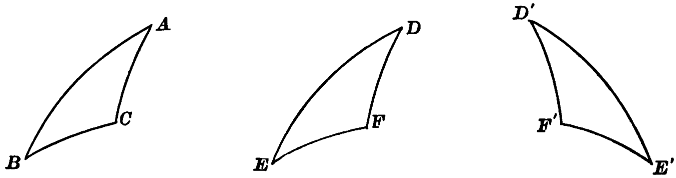
(The proof is left to the pupil; compare § 600.)

PROP. XXII. THEOREM.

602. *If two spherical triangles on the same sphere, or equal spheres, have two sides and the included angle of one equal, respectively, to two sides and the included angle of the other,*

I. *They are equal if the equal parts occur in the same order.*

II. *They are symmetrical if the equal parts occur in the reverse order.*



I. **Given** ABC and DEF spherical ▵ on the same sphere, or equal spheres, having

$$AB = DE,\ AC = DF, \text{ and } \angle A = \angle D;$$

the equal parts occurring in the same order.

To Prove $\triangle ABC = \triangle DEF$.

Proof. Superpose $\triangle ABC$ upon $\triangle DEF$ in such a way that $\angle A$ shall coincide with its equal $\angle D$; side AB falling on side DE, and side AC on side DF.

Then, since $AB = DE$ and $AC = DF$, point B will fall on point E, and point C on point F.

Whence, arc BC will coincide with arc EF. (§ 572)

Hence, ABC and DEF coincide throughout, and are equal.

II. **Given** ABC and $D'E'F'$ spherical ▵ on the same sphere, or equal spheres, having

$$AB = D'E',\ AC = D'F', \text{ and } \angle A = \angle D';$$

the equal parts occurring in the reverse order.

To Prove ABC and $D'E'F'$ symmetrical.

Proof. Let DEF be a spherical $\triangle$ on the same sphere, or an equal sphere, symmetrical to $D'E'F'$, having

$$DE = D'E',\ DF = D'F', \text{ and } \angle D = \angle D';$$

the equal parts occurring in the reverse order.

Then, in spherical ▵ ABC and DEF, we have

$$AB = DE,\ AC = DF, \text{ and } \angle A = \angle D;$$

and the equal parts occur in the same order. (Ax. 1)

$$\therefore \triangle ABC = \triangle DEF. \quad (\S\ 602, \text{I})$$

Therefore, $\triangle ABC$ is symmetrical to $\triangle D'E'F'$.

Prop. XXIII. Theorem.

603. *If two spherical triangles on the same sphere, or equal spheres, have a side and two adjacent angles of one equal, respectively, to a side and two adjacent angles of the other,*

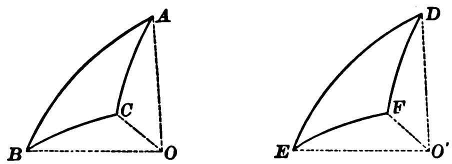
I. *They are equal if the equal parts occur in the same order.*

II. *They are symmetrical if the equal parts occur in the reverse order.*

(The proof is left to the pupil; compare § 602.)

Prop. XXIV. Theorem.

604. *If two spherical triangles on the same sphere, or equal spheres, are mutually equilateral, they are mutually equiangular.*



Given ABC and DEF mutually equilateral spherical ⧍ on equal spheres; sides BC and EF being homologous.

To Prove ABC and DEF mutually equiangular.

Proof. Let O and O' be the centres of the respective spheres, and draw lines OA, OB, OC, $O'D$, $O'E$, and $O'F$.

Now the triedral ∠s O–ABC and O'–DEF have their homologous face ∠s equal. (§ 192)

$\therefore$ diedral $\angle OA =$ diedral $\angle O'D$. (§ 459)

But the $\angle$ between arcs AB and AC is the plane $\angle$ of diedral $\angle OA$, and the $\angle$ between arcs DE and DF is the plane $\angle$ of diedral $\angle O'D$. (§ 586)

$\therefore \angle BAC = \angle EDF$. (§ 434)

In like manner, any two homologous ∠s of ABC and DEF may be proved equal.

Whence, ABC and DEF are mutually equiangular.

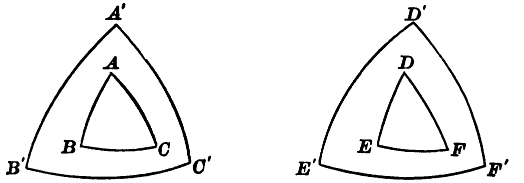
Note. The theorem may be proved in a similar manner when the given spherical ⚠ are on the same sphere.

605. Cor. *If two spherical triangles on the same sphere, or equal spheres, are mutually equilateral,*

1. *They are equal if the equal parts occur in the same order.*
2. *They are symmetrical if the equal parts occur in the reverse order.*

PROP. XXV. THEOREM.

606. *If two spherical triangles on the same sphere, or equal spheres, are mutually equiangular, they are mutually equilateral.*



Given ABC and DEF mutually equiangular spherical ⚠ on the same sphere, or equal spheres.

To Prove ABC and DEF mutually equilateral.

Proof. Let $A'B'C'$ be the polar △ of ABC, and $D'E'F'$ of DEF.

Then, since ABC and DEF are mutually equiangular, $A'B'C'$ and $D'E'F'$ are mutually equilateral. (§ 600)

Then $A'B'C'$ and $D'E'F'$ are mutually equiangular. (§ 604)

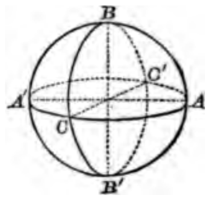
But ABC is the polar △ of $A'B'C'$, and DEF of $D'E'F'$. (§ 592)

Then ABC and DEF are mutually equilateral. (§ 600)

607. Cor. I. *If two spherical triangles on the same sphere, or equal spheres, are mutually equiangular,*

1. *They are equal if the equal parts occur in the same order.*
2. *They are symmetrical if the equal parts occur in the reverse order.*

608. Cor. II. *If three diameters of a sphere be so drawn that each is perpendicular to the other two, the planes determined by them divide the surface of the sphere into eight equal tri-rectangular triangles.*

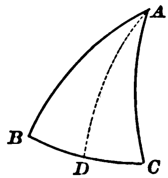


(Prove by § 607, 1. By § 585, each ∠ of each spherical △ is a rt. ∠.)

609. Cor. III. *The surface of a sphere is eight times the surface of one of its tri-rectangular triangles.*

Prop. XXVI. Theorem.

610. *In an isosceles spherical triangle the angles opposite the equal sides are equal.*



Given, in spherical △ ABC, $AB = AC$.

To Prove $\angle B = \angle C$.

Proof. Draw AD an arc of a great ⊙, bisecting side BC at D.

In spherical △s ABD and ACD, $AD = AD$.

Also, $AB = AC$ and $BD = CD$.

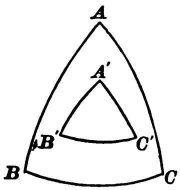
Then, ABD and ACD are mutually equiangular. (§ 604)

$\therefore \angle B = \angle C$.

611. Cor. I. *An isosceles spherical triangle is equal to the spherical triangle which is symmetrical to it.*

For the equal parts occur in the same order.

612. Cor. II. (Converse of Prop. XXVI.) *If two angles of a spherical triangle are equal, the sides opposite are equal.*



Given, in spherical $\triangle ABC$, $\angle B = \angle C$.

To Prove $AB = AC$.

Proof. Let $A'B'C'$ be the polar $\triangle$ of ABC; B being the pole of $A'C'$, and C of $A'B'$.

Then, $A'C'$ is the sup. of $\angle B$, and $A'B'$ of $\angle C$. (§ 593)

$\therefore A'C' = A'B'$. (§ 31, 2)

$\therefore \angle B' = \angle C'$. (§ 610)

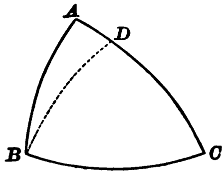
But ABC is the polar $\triangle$ of $A'B'C'$; B' being the pole of AC, and C' of AB. (§ 592)

Then AB is the sup. of $\angle C'$, and AC of $\angle B'$. (?)

$\therefore AB = AC$. (?)

Prop. XXVII. Theorem.

613. *If two angles of a spherical triangle are unequal, the sides opposite are unequal, and the greater side lies opposite the greater angle.*



Given, in spherical $\triangle ABC$, $\angle ABC > \angle C$.

To Prove $AC > AB$.

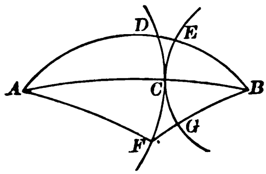
(Prove by a method analogous to that of § 99. Draw BD an arc of a great ⊙ meeting AC at D, and making $\angle CBD$ equal to $\angle C$.)

614. **Cor**. (Converse of Prop. XXVII.) *If two sides of a spherical triangle are unequal, the angles opposite are unequal, and the greater angle lies opposite the greater side.*

(Prove by *Reductio ad Absurdum.*)

PROP. XXVIII. THEOREM.

615. *The shortest line on the surface of a sphere between two given points is the arc of a great circle, not greater than a semi-circumference, which joins the points.*



Given points A and B on the surface of a sphere, and AB an arc of a great ⊙, not > a semi-circumference.

To Prove AB the shortest line on the surface of the sphere between A and B.

Proof. Let C be any point in arc AB.

Let DCF and ECG be arcs of small ⓢ with A and B, respectively, as poles, and AC and BC as polar distances.

Now arcs DCF and ECG have only point C in common.

For let F be any other point in arc DCF, and draw arcs of great ⓢ AF and BF.

$$\therefore AF = AC. \qquad (\S\ 575)$$

But, $$AF + BF > AC + BC. \qquad (\S\ 594)$$

Subtracting arc AF from the first member of the inequality, and its equal arc AC from the second member,

$$BF > BC, \text{ or } BF > BG. \qquad (\S\ 575)$$

Whence, F lies without small ⊙ ECG, and arcs DCF and ECG have only point C in common.

We will next prove that the shortest line on the surface of the sphere from A to B must pass through C.

Let $ADEB$ be any line on the surface of the sphere between A and B, not passing through C, and cutting arcs DCF and ECG at D and E, respectively.

Then, whatever the nature of line AD, it is evident that an equal line can be drawn from A to C.

In like manner, whatever the nature of line BE, an equal line can be drawn from B to C.

Hence, a line can be drawn from A to B passing through C, equal to the sum of lines AD and BE, and consequently $<$ line $ADEB$ by the portion DE.

Therefore, no line which does not pass through C can be the shortest line between A and B.

But by hyp., C is *any* point in arc AB.

Hence, the shortest line from A to B must pass through *every* point of AB.

Then, the arc of a great ⊙ AB is the shortest line on the surface of the sphere between A and B.

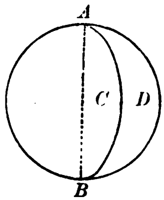
EXERCISES.

2. If the sides of a spherical triangle are 77°, 123°, and 95°, how many degrees are there in each angle of its polar triangle?

3. If the angles of a spherical triangle are 86°, 131°, and 68°, how many degrees are there in each side of its polar triangle?

MEASUREMENT OF SPHERICAL POLYGONS.

DEFINITIONS.



616. A *lune* is a portion of the surface of a sphere bounded by two semi-circumferences of great circles; as $ACBD$.

The *angle* of the lune is the angle between its bounding arcs.

617. A *spherical wedge* is a solid bounded by a lune and the planes of its bounding arcs.

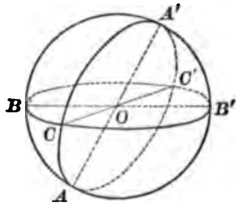
The lune is called the *base* of the spherical wedge.

618. It is evident that *two lunes on the same sphere, or equal spheres, are equal when their angles are equal.*

619. It is evident that *two spherical wedges in the same sphere, or equal spheres, are equal when the angles of the lunes which form their bases are equal.*

PROP. XXIX. THEOREM.

620. *The spherical triangles corresponding to a pair of vertical triedral angles are symmetrical.*



Given AOA', BOB', and COC' diameters of sphere AC; also, the planes determined by them, intersecting the surface in circumferences $ABA'B'$, $BCB'C'$, and $CAC'A'$.

To Prove spherical ▵ ABC and $A'B'C'$ symmetrical.

Proof. ∠s AOB, BOC, and COA are equal, respectively, to ∠s $A'OB'$, $B'OC'$, and $C'OA'$. (§ 40)

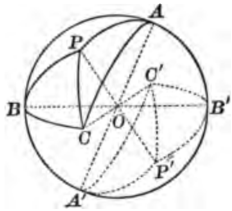
Then, $AB = A'B'$, $BC = B'C'$, and $CA = C'A'$. (§ 192)

But the equal parts of ABC and $A'B'C'$ occur *in the reverse order.*

Whence, ABC and $A'B'C'$ are symmetrical. (§ 605, 2)

PROP. XXX. THEOREM.

621. *Two spherical triangles corresponding to a pair of vertical triedral angles are equivalent.*



Given AOA', BOB', and COC' diameters of sphere AB; also, the planes determined by them, intersecting the surface in arcs AB, BC, CA, $A'B'$, $B'C'$, and $C'A'$.

To Prove area $ABC =$ area $A'B'C'$.

Proof. Let P be the pole of the small ⊙ passing through points A, B, and C, and draw arcs of great ⓢ PA, PB, and PC.

$$\therefore PA = PB = PC. \qquad (\S\ 575)$$

Draw the diameter of the sphere PP', and the arcs of great ⓢ $P'A'$, $P'B'$, and $P'C'$; then, spherical $\triangle$s PAB and $P'A'B'$ are symmetrical. (§ 620)

But spherical $\triangle\, PAB$ is isosceles.

$$\therefore \triangle\, PAB = \triangle\, P'A'B'. \qquad (\S\ 611)$$

In like manner,

$$\triangle\, PBC = \triangle\, P'B'C' \text{ and } \triangle\, PCA = \triangle\, P'C'A'.$$

Then the sum of the areas of $\triangle$s PAB, PBC, and PCA equals the sum of the areas of $P'A'B'$, $P'B'C'$, and $P'C'A'$.

$$\therefore \text{area } ABC = \text{area } A'B'C'.$$

622. Sch. If P and P' fall without spherical $\triangle$s ABC and $A'B'C'$, we should take the sum of the areas of two isosceles spherical $\triangle$s, diminished by the area of a third.

623. Cor. I. *Two symmetrical spherical triangles are equivalent.*

624. Cor. II. Spherical pyramids $O\text{–}APB$, $O\text{–}BPC$, and $O\text{–}CPA$ are equal, respectively, to spherical pyramids $O\text{–}A'P'B'$, $O\text{–}B'P'C'$, and $O\text{–}C'P'A'$. (§ 589)

$$\therefore \text{vol. } O\text{–}ABC = \text{vol. } O\text{–}A'B'C'.$$

Whence, ***the spherical pyramids corresponding to a pair of vertical triedral angles are equivalent.***

EXERCISES.

4. The sum of the angles of a spherical hexagon is greater than 8, and less than 12, right angles. (§ 596.)

5. The sum of the angles of a spherical polygon of n sides is greater than $2n - 4$, and less than $2n$, right angles.

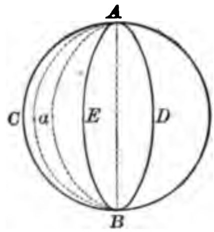
6. The arc of a great circle drawn from the vertex of an isosceles spherical triangle to the middle point of the base, is perpendicular to the base, and bisects the vertical angle.

PROP. XXXI. THEOREM.

625. *Two lunes on the same sphere, or equal spheres, are to each other as their angles.*

Note. The word "*lune*," in the above statement, signifies the *area* of the lune.

Case I. *When the angles are commensurable.*



Given $ACBD$ and $ACBE$ lunes on sphere AB, having their $\angle s$ CAD and CAE commensurable.

To Prove $$\frac{ACBD}{ACBE} = \frac{\angle CAD}{\angle CAE}.$$

Proof. Let $\angle CAa$ be a common measure of $\angle s$ CAD and CAE, and let it be contained 5 times in $\angle CAD$, and 3 times in $\angle CAE$.

$$\therefore \frac{\angle CAD}{\angle CAE} = \frac{5}{3}. \quad (1)$$

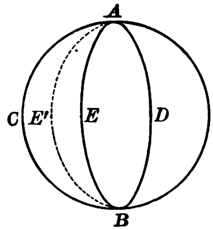
Producing the arcs of division of $\angle CAD$ to B, lune $ACBD$ will be divided into 5 parts, and lune $ACBE$ into 3 parts, all of which parts will be equal. (§ 618)

$$\therefore \frac{ACBD}{ACBE} = \frac{5}{3}. \quad (2)$$

From (1) and (2), $$\frac{ACBD}{ACBE} = \frac{\angle CAD}{\angle CAE}. \quad (?)$$

Note. The theorem may be proved in a similar manner when the given lunes are on equal spheres.

Case II. *When the angles are incommensurable.*



(Prove as in §§ 189 or 244. Let $\angle CAD$ be divided into any number of equal parts, and apply one of these parts to $\angle CAE$ as a unit of measure.)

626. Cor. I. *The surface of a lune is to the surface of the sphere as the angle of the lune is to four right angles.*

For the surface of a sphere may be regarded as a lune whose $\angle$ is equal to 4 rt. $\angle$s.

627. Cor. II. *If the unit of measure for angles is the right angle, the area of a lune is equal to twice its angle, multiplied by the area of a tri-rectangular triangle.*

Given L the area of a lune; A the numerical measure of its $\angle$ referred to a rt. $\angle$ as the unit of measure; and T the area of a tri-rectangular $\triangle$.

To Prove $L = 2\,A \times T.$

Proof. The area of the surface of the sphere is $8\,T$. (§ 609)

$$\therefore \frac{L}{8\,T} = \frac{A}{4}. \qquad (§\ 625)$$

$$\therefore L = \frac{A}{4} \times 8\,T = 2\,A \times T.$$

628. Sch. I. Let it be required to find the area of a lune whose $\angle$ is 50°, on a sphere the area of whose surface is 72.

The $\angle$ of the lune referred to a rt. $\angle$ as the unit of measure is $\frac{5}{9}$; and T is $\frac{1}{8}$ of 72, or 9.

Then the area of the lune is $2 \times \frac{5}{9} \times 9$, or 10.

629. Def. A *tri-rectangular pyramid* is a spherical pyramid whose base is a tri-rectangular triangle.

630. Sch. II. It may be proved, as in § 625, that

Two spherical wedges in the same sphere, or equal spheres, are to each other as the angles of the lunes which form their bases.

(The proof is left to the pupil; see § 619.)

631. Sch. III. It may be proved that

If the unit of measure for angles is the right angle, the volume of a spherical wedge is equal to twice the angle of the lune which forms its base, multiplied by the volume of a tri-rectangular pyramid.

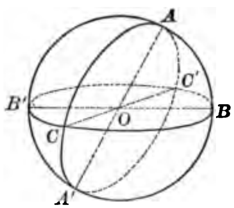
(The proof is left to the pupil; see §§ 626 and 627.)

632. Def. The *spherical excess* of a spherical triangle is the excess of the sum of its angles above 180° (§ 596).

Thus, if the ∡ of a spherical △ are 65°, 80°, and 95°, its spherical excess is $65° + 80° + 95° - 180°$, or 60°.

PROP. XXXII. THEOREM.

633. *If the unit of measure for angles is the right angle, the area of a spherical triangle is equal to its spherical excess, multiplied by the area of a tri-rectangular triangle.*



Given A, B, and C the numerical measures of the ∡ of spherical △ ABC, referred to a rt. ∠ as the unit of measure, and T the area of a tri-rectangular △.

To Prove area $ABC = (A + B + C - 2) \times T$.

Proof. Complete circumferences $ABA'B'$, $ACA'C'$, and $BCB'C'$, and draw diameters AA', BB', and CC'.

Then, since $ABA'C$ is a lune whose $\angle$ is A, we have

$$\text{area } ABC + \text{area } A'BC = 2\,A \times T \text{ (§ 626).} \quad (1)$$

And since $BAB'C$ is a lune whose $\angle$ is B,

$$\text{area } ABC + \text{area } AB'C = 2\,B \times T. \quad (2)$$

Again, $\text{area } A'B'C = \text{area } ABC'$. (§ 620)

Adding area ABC to both members, we have

$$\text{area } ABC + \text{area } A'B'C = \text{area of lune } CBC'A$$
$$= 2\,C \times T. \quad (3)$$

Adding (1), (2), and (3), and observing that the sum of the areas of △ ABC, $A'BC$, $AB'C$, and $A'B'C$ is equal to the area of the surface of a hemisphere, or $4\,T$, we have

$$2 \text{ area } ABC + 4\,T = (2\,A + 2\,B + 2\,C) \times T.$$
$$\therefore \text{ area } ABC + 2\,T = (A + B + C) \times T.$$
$$\therefore \text{ area } ABC = (A + B + C - 2) \times T.$$

634. Sch. I. Let it be required to find the area of a spherical △ whose ∠s are 105°, 80°, and 95°, on a sphere the area of whose surface is 144.

The spherical excess of the spherical △ is 100°, or $\frac{10}{9}$ referred to a rt. $\angle$ as the unit of measure; and the area of a tri-rectangular △ is $\frac{1}{8}$ of 144, or 18.

Then the area of the spherical △ is $\frac{10}{9} \times 18$, or 20.

635. Sch. II. It may be proved, as in § 633, that

If the unit of measure for angles is the right angle, the volume of a triangular spherical pyramid is equal to the spherical excess of its base, multiplied by the volume of a tri-rectangular pyramid.

(The proof is left to the pupil; see §§ 624 and 630.)

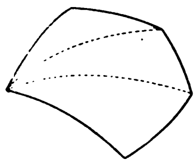
EXERCISES.

7. What is the volume of a spherical wedge the angle of whose base is 127° 30′, if the volume of the sphere is 112?

8. In figure of Prop. XVII., prove $A' = 180° - a$.

PROP. XXXIII. THEOREM.

636. *If the unit of measure for angles is the right angle, the area of any spherical polygon is equal to the sum of its angles, diminished by as many times two right angles as the figure has sides less two, multiplied by the area of a tri-rectangular triangle.*



Given K the area of any spherical polygon, n the number of its sides, s the sum of its ∠s referred to a rt. ∠ as the unit of measure, and T the area of a tri-rectangular △.

To Prove $K = [s - 2(n - 2)] \times T.$

Proof. The spherical polygon can be divided into $n - 2$ spherical △s by drawing diagonals from any vertex.

Now, if the unit of measure for ∠s is the rt. ∠, the area of each spherical △ is equal to the sum of its ∠s, less 2 rt. ∠s, multiplied by T. (§ 633)

Hence, if the unit of measure for ∠s is the rt. ∠, the sum of the areas of the spherical △s is equal to the sum of their ∠s, diminished by $n - 2$ times 2 rt. ∠s, multiplied by T.

But the sum of the ∠s of the spherical △s is equal to the sum of the ∠s of the spherical polygon.

Whence, $K = [s - 2(n - 2)] \times T.$

637. Sch. It may be proved, as in § 636, that

If the unit of measure for angles is the right angle, the volume of any spherical pyramid is equal to the sum of the angles of its base, diminished by as many times two right angles as the base has sides less two, multiplied by the volume of a tri-rectangular pyramid.

(The proof is left to the pupil.)

EXERCISES.

9. The area of a lune is $28\frac{4}{5}$. If the area of the surface of the sphere is 120, what is the angle of the lune?

10. Find the area of a spherical triangle whose angles are 103°, 112°, and 127°, on a sphere the area of whose surface is 160.

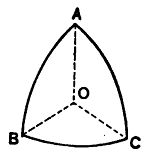
11. Find the volume of a triangular spherical pyramid the angles of whose base are 92°, 119°, and 134°; the volume of the sphere being 192.

12. What is the ratio of the areas of two spherical triangles on the same sphere whose angles are 94°, 135°, and 146°, and 87°, 105°, and 118°, respectively?

13. The area of a spherical triangle, two of whose angles are 78° and 99°, is $34\frac{1}{8}$. If the area of the surface of the sphere is 234, what is the other angle?

14. The volume of a triangular spherical pyramid, the angles of whose base are 105°, 126°, and 147°, is $60\frac{1}{2}$; what is the volume of the sphere?

15. The sides opposite the equal angles of a bi-rectangular triangle are quadrants. (§ 442.)



16. The sides of a spherical triangle, on a sphere the area of whose surface is 156, are 44°, 63°, and 97°. Find the area of its polar triangle.

17. Find the area of a spherical hexagon whose angles are 120°, 139°, 148°, 155°, 162°, and 167°, on a sphere the area of whose surface is 280.

18. Find the volume of a pentagonal spherical pyramid the angles of whose base are 109°, 128°, 137°, 153°, and 158°; the volume of the sphere being 180.

19. The volume of a quadrangular spherical pyramid, the angles of whose base are 110°, 122°, 135°, and 146°, is $12\frac{3}{4}$; what is the volume of the sphere?

20. The area of a spherical pentagon, four of whose angles are 112°, 131°, 138°, and 168°, is 27. If the area of the surface of the sphere is 120, what is the other angle?

21. If two straight lines are tangent to a sphere at the same point, their plane is tangent to the sphere. (§ 400.)

22. The sum of the arcs of great circles drawn from any point within a spherical triangle to the extremities of any side, is less than the sum of the other two sides of the triangle.

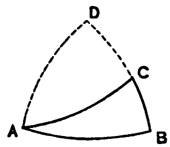
(Compare § 48.)

23. How many degrees are there in the polar distance of a circle, whose plane is $5\sqrt{2}$ units from the centre of the sphere, the diameter of the sphere being 20 units?

(The radius of the ⊙ is a leg of a rt. △, whose hypotenuse is the radius of the sphere, and whose other leg is the distance from its centre to the plane of the ⊙.)

24. The chord of the polar distance of a circle of a sphere is 6. If the radius of the sphere is 5, what is the radius of the circle?

25. If side AB of spherical triangle ABC is a quadrant, and side BC less than a quadrant, prove $\angle A$ less than 90°.

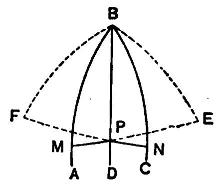


26. The polar distance of a circle of a sphere is 60°. If the diameter of the circle is 6, find the diameter of the sphere, and the distance of the circle from its centre.

(Represent radius of sphere by $2x$.)

27. Any point in the arc of a great circle bisecting a spherical angle is equally distant (§ 573) from the sides of the angle.

(To prove arc PM = arc PN. Let E be a pole of arc AB, and F of arc BC. Spherical ⚠ BPE and BPF are symmetrical by § 602, II., and $PE = PF$.)



28. A point on the surface of a sphere, equally distant from the sides of a spherical angle, lies in the arc of a great circle bisecting the angle.

(Fig. of Ex. 27. To prove $\angle ABP = \angle CBP$. Spherical ⚠ BPE and BPF are symmetrical by § 605, 2.)

29. The arcs of great circles bisecting the angles of a spherical triangle meet in a point equally distant from the sides of the triangle. (Exs. 27, 28, p. 358.)

30. A circle may be inscribed in any spherical triangle.

31. State and prove the theorem for spherical triangles analogous to Prop. IX., I., Book I.

32. State and prove the theorem for spherical triangles analogous to Prop. V., Book I.

33. State and prove the theorem for spherical triangles analogous to Prop. L., Book I. (Ex. 32.)

34. If PA, PB, and PC are three equal arcs of great circles drawn from point P to the circumference of great circle ABC, prove P a pole of ABC.

(PA and PB are quadrants by Ex. 15, p. 357.)

35. The spherical polygons corresponding to a pair of vertical polyedral angles are symmetrical. (§ 456.)

36. A sphere may be inscribed in, or circumscribed about, any tetraedron. (Ex. 73, Book VII.)

37. What is the locus of points in space at a given distance from a given straight line ?

38. Equal small circles of a sphere are equally distant from the centre.

39. State and prove the converse of Ex. 38.

40. The less of two small circles of a sphere is at the greater distance from the centre.

41. State and prove the converse of Ex. 40.

42. What is the locus of points on the surface of a sphere equally distant from the sides of a spherical angle ?

43. If two spheres are tangent to the same plane at the same point, the straight line joining their centres passes through the point of contact.

44. The distance between the centres of two spheres whose radii are 25 and 17, respectively, is 28. Find the diameter of their circle of intersection, and its distance from the centre of each sphere.

45. If a polyedron be circumscribed about each of two equal spheres, the volumes of the polyedrons are to each other as the areas of their surfaces.

(Find the volume of each polyedron by dividing it into pyramids.)

46. Either angle of a spherical triangle is greater than the difference between 180° and the sum of the other two angles.

(Fig. of Prop. XX. To prove $\angle A > 180° - (\angle B + \angle C)$, or $> (\angle B + \angle C) - 180°$, according as $\angle B + \angle C$ is $<$ or $>$ 180°. In the latter case, $A'C' + A'B' > B'C'$; then use § 593.)

Book IX.

MEASUREMENT OF THE CYLINDER, CONE, AND SPHERE.

THE CYLINDER.

Definitions.

638. The *lateral area* of a cylinder is the area of its lateral surface.

A *right section* of a cylinder is a section made by a plane perpendicular to the elements of its lateral surface.

639. A prism is said to be *inscribed in a cylinder* when its lateral edges are elements of the cylindrical surface.

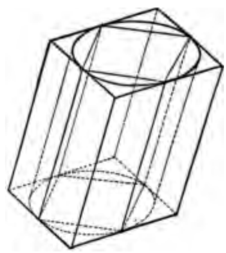
In this case, the bases of the prism are inscribed in the bases of the cylinder.

A prism is said to be *circumscribed about a cylinder* when its lateral faces are tangent to the cylinder, and its bases lie in the same planes with the bases of the cylinder.

In this case, the bases of the prism are circumscribed about the bases of the cylinder.

640. It follows from § 363 that

If a prism whose base is a regular polygon be inscribed in, or circumscribed about, a circular cylinder (§ 540), *and the number of its faces be indefinitely increased,*



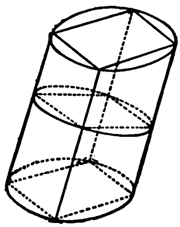
1. *The lateral area of the prism approaches the lateral area of the cylinder as a limit.*

2. *The volume of the prism approaches the volume of the cylinder as a limit.*

3. *The perimeter of a right section of the prism approaches the perimeter of a right section of the cylinder as a limit.**

PROP. I. THEOREM.

641. *The lateral area of a circular cylinder is equal to the perimeter of a right section multiplied by an element of the lateral surface.*



Given S the lateral area, P the perimeter of a rt. section, and E an element of the lateral surface, of a circular cylinder.

To Prove $S = P \times E$.

Proof. Inscribe in the cylinder a prism whose base is a regular polygon, and let S' denote its lateral area, and P' the perimeter of a rt. section.

Then, since the lateral edge of the prism is E,

$$S' = P' \times E. \qquad (\S\ 484)$$

Now let the number of faces of the prism be indefinitely increased.

Then, S' approaches the limit S,

and $P' \times E$ approaches the limit $P \times E$. (§ 640, 1, 3)

By the Theorem of Limits, these limits are equal. (§ 188)

$$\therefore\ S = P \times E.$$

* For rigorous proofs of these statements, see Appendix, p. 386.

642. Cor. I. *The lateral area of a cylinder of revolution is equal to the circumference of its base multiplied by its altitude.*

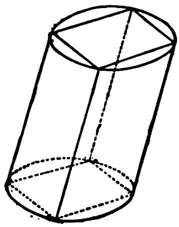
643. Cor. II. If S denotes the lateral area, T the total area, H the altitude, and R the radius of the base, of a cylinder of revolution,

$$S = 2\pi RH. \qquad (§\ 368)$$

And, $T = 2\pi RH + 2\pi R^2$ (§ 371) $= 2\pi R(H + R)$.

PROP. II. THEOREM.

644. *The volume of a circular cylinder is equal to the product of its base and altitude.*



Given V the volume, B the area of the base, and H the altitude, of a circular cylinder.

To Prove $V = B \times H$.

Proof. Inscribe in the cylinder a prism whose base is a regular polygon, and let V' denote its volume, and B' the area of its base.

Then, since the altitude of the prism is H,

$$V' = B' \times H. \qquad (§\ 499)$$

Now let the number of faces of the prism be indefinitely increased.

Then, V' approaches the limit V. (§ 640, 2)

And, $B' \times H$ approaches the limit $B \times H$. (§ 363, II)

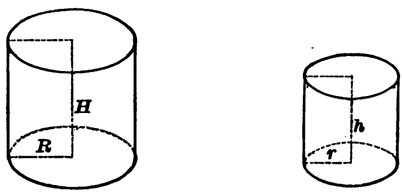
$$\therefore\ V = B \times H. \qquad (?)$$

645. Cor. If V denotes the volume, H the altitude, and R the radius of the base, of a circular cylinder,

$$V = \pi R^2 H. \qquad (?)$$

Prop. III. Theorem.

646. *The lateral or total areas of two similar cylinders of revolution* (§ 550) *are to each other as the squares of their altitudes, or as the squares of the radii of their bases; and their volumes are to each other as the cubes of their altitudes, or as the cubes of the radii of their bases.*



Given S and s the lateral areas, T and t the total areas, V and v the volumes, H and h the altitudes, and R and r the radii of the bases, of two similar cylinders of revolution.

To Prove $\frac{S}{s} = \frac{T}{t} = \frac{H^2}{h^2} = \frac{R^2}{r^2}$, and $\frac{V}{v} = \frac{H^3}{h^3} = \frac{R^3}{r^3}$.

Proof. Since the generating rectangles are similar,

$$\frac{H}{h} = \frac{R}{r} \qquad (§ 253, 2)$$

$$= \frac{H + R}{h + r}. \qquad (§ 240)$$

$$\therefore \frac{S}{s} = \frac{2\pi RH}{2\pi rh} \quad (§ 643) = \frac{R}{r} \times \frac{R}{r} = \frac{R^2}{r^2} = \frac{H^2}{h^2},$$

$$\frac{T}{t} = \frac{2\pi R(H + R)}{2\pi r(h + r)} \quad (§ 643) = \frac{R}{r} \times \frac{R}{r} = \frac{R^2}{r^2} = \frac{H^2}{h^2},$$

and $$\frac{V}{v} = \frac{\pi R^2 H}{\pi r^2 h} \quad (§ 645) = \frac{R^2}{r^2} \times \frac{R}{r} = \frac{R^3}{r^3} = \frac{H^3}{h^3}.$$

EXERCISES.

1. Find the lateral area, total area, and volume of a cylinder of revolution, the diameter of whose base is 18, and whose altitude is 16.

2. The radii of the bases of two similar cylinders of revolution are 24 and 44, respectively. If the lateral area of the first cylinder is 720, what is the lateral area of the second?

3. Find the altitude and diameter of the base of a cylinder of revolution, whose lateral area is 168π and volume 504π.

(Substitute the given values in the formulæ of §§ 643 and 645, and solve the resulting equations.)

4. Find the volume of a cylinder of revolution, whose total area is 170π and altitude 12.

5. How many cubic feet of metal are there in a hollow cylindrical tube 18 ft. long, whose outer diameter is 8 in., and thickness 1 in.?

(Find the difference of the volumes of two cylinders of revolution. $\pi = 3.1416$.)

6. The cross-section of a tunnel, $2\frac{1}{2}$ miles in length, is in the form of a rectangle 6 yd. wide and 4 yd. high, surmounted by a semicircle whose diameter is equal to the width of the rectangle; how many cu. yd. of material were taken out in its construction? ($\pi = 3.1416$.)

7. The volume of a cylinder of revolution is equal to its lateral area multiplied by one-half the radius of its base.

THE CONE.

DEFINITIONS.

647. The *lateral area* of a cone, or frustum of a cone, is the area of its lateral surface.

The *slant height* of a cone of revolution is the straight line drawn from the vertex to any point in the circumference of the base.

The *slant height* of a frustum of a cone of revolution is that portion of the slant height of the cone included between the bases of the frustum.

648. A pyramid is said to be *inscribed in a cone* when its lateral edges are elements of the conical surface; the base of the pyramid is inscribed in the base of the cone, and its vertex coincides with the vertex of the cone.

A pyramid is said to be *circumscribed about a cone* when its lateral faces are tangent to the cone, and its base lies in the same plane with the base of the cone; the base of the pyramid is circumscribed about the base of the cone, and its vertex coincides with the vertex of the cone.

649. A frustum of a pyramid is said to be *inscribed in a frustum of a cone* when its lateral edges are elements of the lateral surface of the frustum of the cone.

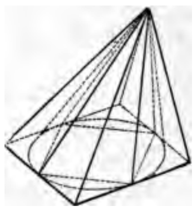
In this case, the bases of the frustum of the pyramid are inscribed in the bases of the frustum of the cone.

A frustum of a pyramid is said to be *circumscribed about a frustum of a cone* when its lateral faces are tangent to the frustum of the cone, and its bases lie in the same planes with the bases of the frustum of the cone.

In this case, the bases of the frustum of the pyramid are circumscribed about the bases of the frustum of the cone.

650. It follows from § 363 that

If a pyramid whose base is a regular polygon be inscribed in, or circumscribed about, a circular cone (§ 553), *and the number of its faces be indefinitely increased,*



1. *The lateral area of the pyramid approaches the lateral area of the cone as a limit.*

2. *The volume of the pyramid approaches the volume of the cone as a limit.**

651. It follows from the above that

If a frustum of a pyramid whose base is a regular polygon be inscribed in, or circumscribed about, a frustum of a circular cone, and the number of its faces be indefinitely increased,

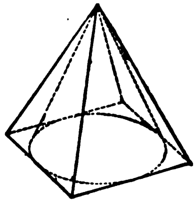
1. *The lateral area of the frustum of the pyramid approaches the lateral area of the frustum of the cone as a limit.*

2. *The volume of the frustum of the pyramid approaches the volume of the frustum of the cone as a limit.*

* For rigorous proofs of these statements, see Appendix, p. 388.

PROP. IV. THEOREM.

652. *The lateral area of a cone of revolution is equal to the circumference of its base, multiplied by one-half its slant height.*



Given S the lateral area, C the circumference of the base, and L the slant height, of a cone of revolution.

To Prove $S = C \times \frac{1}{2} L.$

Proof. Circumscribe about the cone a regular pyramid; let S' denote its lateral area, and C' the perimeter of its base.

Now the sides of the base of the pyramid are bisected at their points of contact with the base of the cone. (§ 174)

Then, the slant height of the pyramid is the same as the slant height of the cone. (§ 508)

$\therefore S' = C' \times \frac{1}{2} L.$ (§ 512)

Now let the number of faces of the pyramid be indefinitely increased.

Then, S' approaches the limit S. (§ 650, 1)

And $C' \times \frac{1}{2} L$ approaches the limit $C \times \frac{1}{2} L$. (§ 363, I)

$\therefore S = C \times \frac{1}{2} L.$ (?)

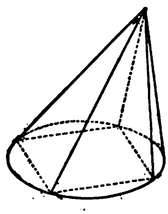
653. Cor. If S denotes the lateral area, T the total area, L the slant height, and R the radius of the base, of a cone of revolution,

$$S = 2\pi R \times \tfrac{1}{2} L \text{ (?) } = \pi RL.$$

And, $T = \pi RL + \pi R^2 \text{ (?) } = \pi R(L + R).$

PROP. V. THEOREM.

654. *The volume of a circular cone is equal to the area of its base, multiplied by one-third its altitude.*



Given V the volume, B the area of the base, and H the altitude, of a circular cone.

To Prove $V = B \times \frac{1}{3} H.$

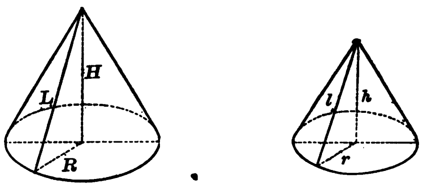
(Inscribe a pyramid whose base is a regular polygon.)

655. Cor. If V denotes the volume, H the altitude, and R the radius of the base, of a circular cone,

$$V = \frac{1}{3} \pi R^2 H. \qquad (?)$$

PROP. VI. THEOREM.

656. *The lateral or total areas of two similar cones of revolution are to each other as the squares of their slant heights, or as the squares of their altitudes, or as the squares of the radii of their bases; and their volumes are to each other as the cubes of their slant heights, or as the cubes of their altitudes, or as cubes of the radii of their bases.*



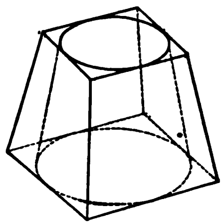
Given S and s the lateral areas, T and t the total areas, V and v the volumes, L and l the slant heights, H and h the altitudes, and R and r the radii of the bases, of two similar cones of revolution (§ 555).

To Prove $\frac{S}{s} = \frac{T}{t} = \frac{L^2}{l^2} = \frac{H^2}{h^2} = \frac{R^2}{r^2}$, and $\frac{V}{v} = \frac{L^3}{l^3} = \frac{H^3}{h^3} = \frac{R^3}{r^3}$.

(The proof is left to the pupil; compare § 646.)

PROP. VII. THEOREM.

657. *The lateral area of a frustum of a cone of revolution is equal to the sum of the circumferences of its bases, multiplied by one-half its slant height.*



Given S the lateral area, C and c the circumferences of the bases, and L the slant height, of a frustum of a cone of revolution,

To Prove $S = (C + c) \times \frac{1}{2} L.$

Proof. Circumscribe about the frustum of the cone a frustum of a regular pyramid; let S' denote its lateral area, and C' and c' the perimeters of its bases.

Now the sides of the bases of the frustum of the pyramid are bisected at their points of contact with the bases of the frustum of the cone. (§ 174)

Then, the slant height of the frustum of the pyramid is the same as the slant height of the frustum of the cone. (§ 508)

$\therefore S' = (C' + c') \times \frac{1}{2} L.$ (§ 513)

Now let the number of faces of the frustum of the pyramid be indefinitely increased.

Then, S' approaches the limit S, (§ 651, 1)

and $(C' + c') \times \frac{1}{2} L$ approaches the limit $(C + c) \times \frac{1}{2} L.$ (§ 363, I)

$\therefore S = (C + c) \times \frac{1}{2} L.$ (?)

658. Cor. I. If S denotes the lateral area, L the slant height, and R and r the radii of the bases, of a frustum of a cone of revolution,

$$S = (2\pi R + 2\pi r) \times \frac{1}{2}L\,(?) = \pi(R + r)L.$$

659. Cor. II. We may write the first result of § 658

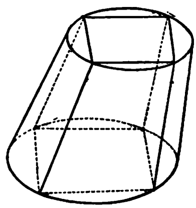
$$S = 2\pi \times \frac{1}{2}(R + r) \times L.$$

But, $2\pi \times \frac{1}{2}(R + r)$ is the circumference of a section equally distant from the bases. (§ 132)

Whence, *the lateral area of a frustum of a cone of revolution is equal to the circumference of a section equally distant from its bases, multiplied by its slant height.*

PROP. VIII. THEOREM.

660. *The volume of a frustum of a circular cone is equal to the sum of its bases and a mean proportional between its bases, multiplied by one-third its altitude.*



Given V the volume, B and b the areas of the bases, and H the altitude, of a frustum of a circular cone.

To Prove $V = (B + b + \sqrt{B \times b}) \times \frac{1}{3}H.$

(Inscribe a frustum of a pyramid whose base is a regular polygon. Then apply § 524.)

661. Cor. If V denotes the volume, H the altitude, and R and r the radii of the bases, of a frustum of a circular cone,

$$B = \pi R^2,\ b = \pi r^2, \text{ and } \sqrt{B \times b} = \sqrt{\pi^2 R^2 r^2} = \pi R r. \quad (?)$$

Then,

$$V = (\pi R^2 + \pi r^2 + \pi R r) \times \frac{1}{3}H = \frac{1}{3}\pi(R^2 + r^2 + Rr)H.$$

EXERCISES.

8. Find the lateral area, total area, and volume of a cone of revolution, the radius of whose base is 7, and whose slant height is 25.

9. Find the lateral area, total area, and volume of a frustum of a cone of revolution, the diameters of whose bases are 16 and 6, and whose altitude is 12.

10. The slant heights of two similar cones of revolution are 9 and 15, respectively. If the volume of the second cone is 625, what is the volume of the first?

11. Find the volume of a cone of revolution, whose slant height is 29 and lateral area 580π.

12. Find the lateral area of a cone of revolution, whose volume is 320π and altitude 15.

13. The altitude of a cone of revolution is 27, and the radius of its base is 16. What is the diameter of the base of an equivalent cylinder of revolution, whose altitude is 16?

14. The area of the entire surface of a frustum of a cone of revolution is 306π, and the radii of its bases are 11 and 5. Find its lateral area and volume.

15. The volume of a frustum of a cone of revolution is 6020π, its altitude is 60, and the radius of its lower base is 15. Find the radius of its upper base and its lateral area.

16. Find the altitude and lateral area of a cone of revolution, whose volume is 800π, and whose slant height is to the diameter of its base as 13 to 10.

17. The total areas of two similar cylinders of revolution are 32 and 162, respectively. If the volume of the second cylinder is 1458, what is the volume of the first?

(Let x and y denote the altitudes of the cylinders.)

18. The volumes of two similar cones of revolution are 343 and 512, respectively. If the lateral area of the first cone is 196, what is the lateral area of the second?

19. A cubical piece of lead, the area of whose entire surface is 384 sq. in., is melted and formed into a cone of revolution, the radius of whose base is 12 in. Find the altitude of the cone.

20. A tapering hollow iron column, 1 in. thick, is 24 ft. long, 10 in. in outside diameter at one end, and 8 in. in diameter at the other; how many cubic inches of metal were used in its construction?

(Find the difference of the volumes of the frustums of two cones of revolution. $\pi = 3.1416$.)

21. If the altitude of a cone of revolution is three-fourths the radius of its base, its volume is equal to its lateral area multiplied by one-fifth the radius of its base.

THE SPHERE.

DEFINITIONS.

662. A *zone* is a portion of the surface of a sphere included between two parallel planes.

The circumferences of the circles which bound the zone are called the *bases*, and the perpendicular distance between their planes the *altitude*.

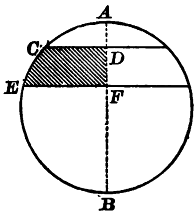
A *zone of one base* is a zone one of whose bounding planes is tangent to the sphere.

A *spherical segment* is a portion of a sphere included between two parallel planes.

The circles which bound it are called the *bases*, and the perpendicular distance between them the *altitude*.

A *spherical segment of one base* is a spherical segment one of whose bounding planes is tangent to the sphere.

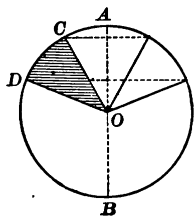
663. If semicircle $ACEB$ be revolved about diameter AB as an axis, and CD and EF are lines $\perp AB$, arc CE generates a zone whose altitude is DF, figure $CEFD$ a spherical segment whose altitude is DF, arc AC a zone of one base, and figure ACD a spherical segment of one base.



664. If a semicircle be revolved about its diameter as an axis, the solid generated by any sector of the semicircle is called a *spherical sector*.

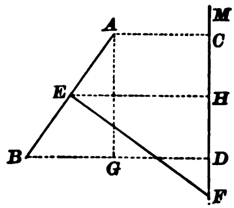
Thus, if semicircle $ACDB$ be revolved about diameter AB as an axis, sector OCD generates a spherical sector.

The zone generated by the arc of the sector is called the *base* of the spherical sector.



PROP. IX. THEOREM.

665. *The area of the surface generated by the revolution of a straight line about a straight line in its plane, not parallel to and not intersecting it, as an axis, is equal to its projection on the axis, multiplied by the circumference of a circle, whose radius is the perpendicular erected at the middle point of the line and terminating in the axis.*



Given str. line AB revolved about str. line FM in its plane, not ∥ to and not intersecting it, as an axis; lines AC and $BD \perp FM$, and EF the ⊥ erected at the middle point of AB terminating in FM.

To Prove area $AB^* = CD \times 2\pi EF$. (§§ 276, 368)

Proof. Draw line $AG \perp BD$, and line $EH \perp CD$.

The surface generated by AB is the lateral surface of a frustum of a cone of revolution, whose bases are generated by AC and BD.

$\therefore$ area $AB = AB \times 2\pi EH$. (§ 659)

But △ ABG and EFH are similar. (§ 262)

$$\therefore \frac{AB}{AG} = \frac{EF}{EH}. \quad (?)$$

$$\therefore AB \times EH = AG \times EF \quad (§ 232)$$

$$= CD \times EF. \quad (?)$$

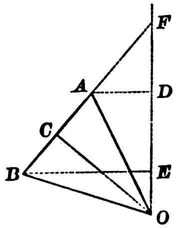
Substituting, we have

$$\text{area } AB = CD \times 2\pi EF.$$

* The expression "area AB" is used to denote *the area of the surface generated by AB.*

PROP. X. THEOREM.

666. *If an isosceles triangle be revolved about a straight line in its plane, not parallel to its base, as an axis, which passes through its vertex without intersecting its surface, the volume of the solid generated is equal to the area of the surface generated by the base, multiplied by one-third the altitude.*



Given isosceles $\triangle\ OAB$ revolved about str. line OF in its plane, not $\parallel$ to base AB, as an axis; and line $OC \perp AB$.

To Prove vol. OAB* $=$ area $AB \times \frac{1}{3}\ OC$.

Proof. Draw lines AD and $BE \perp OF$; and produce BA to meet OF at F.

Now, vol. $OBF =$ vol. $OBE +$ vol. BEF

$$= \tfrac{1}{3}\pi\overline{BE}^2 \times OE + \tfrac{1}{3}\pi\overline{BE}^2 \times EF \qquad (§\ 655)$$

$$= \tfrac{1}{3}\pi\overline{BE}^2 \times (OE + EF) = \tfrac{1}{3}\pi BE \times BE \times OF.$$

But $BE \times OF = OC \times BF$, for each expresses twice the area of $\triangle OBF$. (?)

$$\therefore \text{ vol. } OBF = \tfrac{1}{3}\pi BE \times OC \times BF.$$

But $\pi BE \times BF$ is the area of the surface generated by BF. (§ 653)

$$\therefore \text{ vol. } OBF = \text{area } BF \times \tfrac{1}{3}\ OC. \qquad (1)$$

Similarly, vol. $OAF =$ area $AF \times \frac{1}{3}\ OC$. (2)

Subtracting (2) from (1), we have

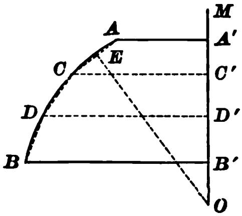
$$\text{vol. } OAB = (\text{area } BF - \text{area } AF) \times \tfrac{1}{3}\ OC$$

$$= \text{area } AB \times \tfrac{1}{3}\ OC.$$

* The expression "vol. OAB" is used to denote *the volume of the solid generated by OAB.*

PROP. XI. THEOREM.

667. *The area of a zone is equal to its altitude multiplied by the circumference of a great circle.*



Given arc AB revolved about diameter OM as an axis, lines AA' and $BB' \perp OM$, and R the radius of the arc.

To Prove area of zone generated by $AB = A'B' \times 2\,\pi R$.

Proof. Divide arc AB into three equal arcs, AC, CD, and DB, and draw chords AC, CD, and DB.

Also, draw lines CC' and $DD' \perp OM$, and line $OE \perp AC$.

$\therefore$ area $AC = A'C' \times 2\,\pi OE$,

area $CD = C'D' \times 2\,\pi OE$, etc. (§ 665)

Adding these equations, we have

area of surface generated by broken line $ACDB$

$= (A'C' + C'D' + \text{etc.}) \times 2\,\pi OE = A'B' \times 2\,\pi OE$.

Now let the subdivisions of arc AB be bisected indefinitely.

Then, area of surface generated by broken line $ACDB$ approaches area of surface generated by arc AB as a limit. (§ 363, I*)

And, $A'B' \times 2\,\pi OE$ approaches $A'B' \times 2\,\pi R$ as a limit. (§ 364, 1*)

* The broken line $ACDB$ is called a *regular broken line*, and is said to be *inscribed in arc AB*; the theorems of §§ 363, I, and 364, 1, are evidently true when, instead of the perimeter of a regular inscribed polygon, we have a regular broken line inscribed in an arc.

For a rigorous proof of the statement that the area of the surface generated by $ACDB$ approaches the area of the surface generated by arc AB as a limit, see Appendix, p. 390.

Then, area of zone generated by arc $AB = A'B' \times 2\,\pi R$.
(§ 188)

668. Sch. The proof of § 667 holds for any zone which lies entirely on the surface of a hemisphere; for, in that case, no chord is ∥ OM, and § 665 is applicable.

Since a zone which does not lie entirely on the surface of a hemisphere may be considered as the sum of two zones, each of which does lie entirely on the surface of a hemisphere, the theorem of § 667 is true for any zone.

669. Cor. I. If S denotes the area of a zone, h its altitude, and R the radius of the sphere,

$$S = 2\,\pi Rh.$$

670. Cor. II. Since the surface of a sphere may be regarded as a zone whose altitude is a diameter of the sphere, it follows that

The area of the surface of a sphere is equal to its diameter multiplied by the circumference of a great circle.

671. Cor. III. Let S denote the area of the surface of a sphere, R its radius, and D its diameter.

Then, $\quad S = 2\,R \times 2\,\pi R\,(?) = 4\,\pi R^2.$

That is, *the area of the surface of a sphere is equal to the square of its radius multiplied by* $4\,\pi$.

Again, $\quad S = \pi \times (2\,R)^2 = \pi D^2.$

That is, *the area of the surface of a sphere is equal to the square of its diameter multiplied by* π.

672. Cor. IV. *The surface of a sphere is equivalent to four great circles.*

For πR^2 is the area of a great ⊙. (?)

673. Cor. V. *The areas of the surfaces of two spheres are to each other as the squares of their radii, or as the squares of their diameters.*

(The proof is left to the pupil; compare § 372.)

EXERCISES.

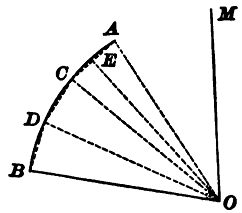
22. Find the area of the surface of a sphere whose radius is 12.

23. Find the area of a zone whose altitude is 13, if the radius of the sphere is 16.

24. Find the area of a spherical triangle whose angles are 125°, 133°, and 156°, on a sphere whose radius is 10.

PROP. XII. THEOREM.

674. *The volume of a spherical sector is equal to the area of the zone which forms its base, multiplied by one-third the radius of the sphere.*



Given sector OAB revolved about diameter OM as an axis, and R the radius of the arc.

To Prove volume of spherical sector generated by OAB = area of zone generated by $AB \times \frac{1}{3} R$.

Proof. Divide arc AB into three equal arcs, AC, CD, and DB, and draw chords AC, CD, and DB.

Also, draw lines OC and OD, and line $OE \perp AC$.

$\therefore$ vol. OAC = area $AC \times \frac{1}{3} OE$,

vol. OCD = area $CD \times \frac{1}{3} OE$, etc. (§ 666)

Adding these equations, we have

volume of solid generated by polygon $OACDB$

= (area AC + area CD + etc.) $\times \frac{1}{3} OE$

= area $ACDB \times \frac{1}{3} OE$.

Now let the subdivisions of arc AB be bisected indefinitely.

Then, volume of solid generated by polygon $OACDB$ approaches volume of solid generated by sector OAB as a limit. (§ 363, II *)

And area of surface generated by $ACDB \times \frac{1}{3} OE$ approaches area of surface generated by arc $AB \times \frac{1}{3} R$ as a limit. (§§ 363, I, 364, 1 †)

Then, volume of solid generated by sector OAB

$$= \text{area of zone generated by arc } AB \times \tfrac{1}{3} R. \qquad (?)$$

675. Sch. It is evident, as in § 668, that the theorem of § 674 holds for any spherical sector.

676. Cor. I. If V denotes the volume of a spherical sector, h the altitude of the zone which forms its base, and R the radius of the sphere,

$$V = 2\pi Rh \times \tfrac{1}{3} R \,(\S\ 669) = \tfrac{2}{3}\pi R^2 h.$$

677. Cor. II. Since a sphere may be regarded as a spherical sector whose base is the surface of the sphere,

The volume of a sphere is equal to the area of its surface multiplied by one-third its radius.

678. Cor. III. Let V denote the volume of a sphere, R its radius, and D its diameter.

Then, $V = 4\pi R^2 \times \frac{1}{3} R \,(\S\ 671) = \frac{4}{3}\pi R^3.$

That is, *the volume of a sphere is equal to the cube of its radius multiplied by* $\frac{4}{3}\pi$.

Again, $V = \pi D^2 \times \frac{1}{6} D \,(\S\ 671) = \frac{1}{6}\pi D^3.$

That is, *the volume of a sphere is equal to the cube of its diameter multiplied by* $\frac{1}{6}\pi$.

* The polygon $OACDB$ is called a *regular polygonal sector*, and is said to be *inscribed in sector* OAB; the theorem of § 363, II, is evidently true when, instead of a regular inscribed polygon, we have a regular polygonal sector inscribed in a sector.

For a rigorous proof of the statement that the volume of the solid generated by $OACDB$ approaches the volume of the solid generated by sector OAB as a limit, see Appendix, p. 391.

† See note foot of p. 374.

679. Cor. IV. *The volumes of two spheres are to each other as the cubes of their radii, or as the cubes of their diameters.*

(The proof is left to the pupil.)

680. Cor. V. *The volume of a spherical pyramid is equal to the area of its base multiplied by one-third the radius of the sphere.*

Given P the volume of a spherical pyramid, K the area of its base, and R the radius of the sphere.

To Prove $P = K \times \frac{1}{3} R.$

Proof. Let n denote the number of sides of the base of the spherical pyramid, s the sum of its ∠s referred to a rt. ∠ as the unit of measure, T the area of a tri-rectangular △, T' the volume of a tri-rectangular pyramid, S the area of the surface of the sphere, and V its volume.

Then, $\frac{P}{K} = \frac{[s - 2(n-2)] \times T'}{[s - 2(n-2)] \times T} = \frac{T'}{T}.$ (§§ 636, 637)

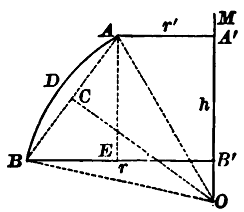
Also, $\frac{V}{S} = \frac{8T'}{8T} = \frac{T'}{T}.$ (§ 609)

$\therefore \frac{P}{K} = \frac{V}{S} = \frac{\frac{4}{3}\pi R^3}{4\pi R^2}$ (§§ 671, 678) $= \frac{1}{3} R.$

$\therefore P = K \times \frac{1}{3} R.$

Prop. XIII. Problem.

681. *Given the radii of the bases, and the altitude, of a spherical segment, to find its volume.*



Given O the centre of arc ADB, lines AA' and $BB' \perp$ to diameter OM, $AA' = r'$, $BB' = r$, $A'B' = h$, and figure $ADBB'A'$ revolved about OM as an axis.

Required to express volume of spherical segment generated by $ADBB'A'$ in terms of r, r', and h.

Solution. Draw lines OA, OB, and AB; also, line $OC \perp AB$, and line $AE \perp BB'$; and denote radius OA by R.

Now, vol. $ADBB'A' =$ vol. $ACBD +$ vol. $ABB'A'$. (1)

Also, vol. $ACBD =$ vol. $OADB -$ vol. OAB.

But, vol. $OADB = \frac{2}{3}\pi R^2 h$. (§ 676)

And, vol. $OAB =$ area $AB \times \frac{1}{3} OC$ (§ 666)

$= h \times 2\pi OC \times \frac{1}{3} OC$ (§ 665)

$= \frac{2}{3}\pi \overline{OC}^2 h$.

$\therefore$ vol. $ACDB = \frac{2}{3}\pi R^2 h - \frac{2}{3}\pi \overline{OC}^2 h$

$= \frac{2}{3}\pi (R^2 - \overline{OC}^2) h$.

But, $R^2 - \overline{OC}^2 = \overline{AC}^2$ (§ 273)

$= (\frac{1}{2} AB)^2$ (?)

$= \frac{1}{4}\overline{AB}^2$.

$\therefore$ vol. $ACDB = \frac{2}{3}\pi \times \frac{1}{4}\overline{AB}^2 \times h = \frac{1}{6}\pi \overline{AB}^2 h$.

Now, $\overline{AB}^2 = \overline{BE}^2 + \overline{AE}^2$ (?)

$= (r - r')^2 + h^2$. (?)

$\therefore$ vol. $ACDB = \frac{1}{6}\pi [(r - r')^2 + h^2] h$.

Also, vol. $ABB'A' = \frac{1}{3}\pi (r^2 + r'^2 + rr') h$. (§ 661)

Substituting in (1), we have

vol. $ADBB'A'$

$= \frac{1}{6}\pi [(r - r')^2 + h^2] h + \frac{1}{6}\pi (2r^2 + 2r'^2 + 2rr') h$

$= \frac{1}{6}\pi (r^2 - 2rr' + r'^2 + h^2 + 2r^2 + 2r'^2 + 2rr') h$

$= \frac{1}{6}\pi (3r^2 + 3r'^2) h + \frac{1}{6}\pi h^3$

$= \frac{1}{2}\pi (r^2 + r'^2) h + \frac{1}{6}\pi h^3$.

682. Cor. If r denotes the radius of the base, and h the altitude, of a spherical segment of one base, its volume is

$$\tfrac{1}{2}\pi r^2 h + \tfrac{1}{6}\pi h^3.$$

EXERCISES.

25. Find the volume of a sphere whose radius is 12.

26. Find the volume of a spherical sector, the altitude of whose base is 12, the diameter of the sphere being 25.

27. Find the volume of a spherical segment, the radii of whose bases are 4 and 5, and whose altitude is 9.

28. Find the radius and volume of a sphere, the area of whose surface is $324\,\pi$.

29. Find the diameter and area of the surface of a sphere whose volume is $\frac{1125}{2}\pi$.

30. The surface of a sphere is equivalent to the lateral surface of its circumscribed cylinder.

31. The volume of a sphere is two-thirds the volume of its circumscribed cylinder.

32. A spherical cannon-ball 9 in. in diameter is dropped into a cubical box filled with water, whose depth is 9 in. How many cubic inches of water will be left in the box? ($\pi = 3.1416$.)

33. What is the angle of the base of a spherical wedge whose volume is $\frac{40}{3}\pi$, if the radius of the sphere is 4?

34. Find the volume of a quadrangular spherical pyramid, the angles of whose base are 107°, 118°, 134°, and 146°; the diameter of the sphere being 12.

35. The surface of a sphere is equivalent to two-thirds the entire surface of its circumscribed cylinder.

36. Prove Prop. IX. when the straight line is parallel to the axis.

37. Find the area of the surface and the volume of a sphere inscribed in a cube the area of whose surface is 486.

38. How many spherical bullets, each $\frac{3}{8}$ in. in diameter, can be formed from five pieces of lead, each in the form of a cone of revolution, the radius of whose base is 5 in., and whose altitude is 8 in.?

39. A cylindrical vessel, 8 in. in diameter, is filled to the brim with water. A ball is immersed in it, displacing water to the depth of $2\frac{1}{4}$ in. Find the diameter of the ball.

40. If a sphere 6 in. in diameter weighs 351 ounces, what is the weight of a sphere of the same material whose diameter is 10 in.?

41. If a sphere whose radius is $12\frac{1}{2}$ in. weighs 3125 lb., what is the radius of a sphere of the same material whose weight is $819\frac{1}{5}$ lb.?

42. The altitude of a frustum of a cone of revolution is $3\frac{1}{2}$, and the radii of its bases are 5 and 3; what is the diameter of an equivalent sphere?

43. Find the radius of a sphere whose surface is equivalent to the entire surface of a cylinder of revolution, whose altitude is $10\frac{1}{2}$, and radius of base 3.

44. The volume of a cylinder of revolution is equal to the area of its generating rectangle, multiplied by the circumference of a circle whose radius is the distance to the axis from the centre of the rectangle.

45. The volume of a cone of revolution is equal to its lateral area, multiplied by one-third the perpendicular from the vertex of the right angle to the hypotenuse of the generating triangle.

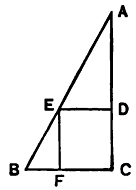
46. Two zones on the same sphere, or equal spheres, are to each other as their altitudes.

47. The area of a zone of one base is equal to the area of the circle whose radius is the chord of its generating arc. (§ 270, 2.)

48. If the radius of a sphere is R, what is the area of a zone of one base, whose generating arc is 45°? (Ex. 55, p. 210.)

49. If the altitude of a cone of revolution is 15, and its slant height 17, find the total area of an inscribed cylinder, the radius of whose base is 5.

(Let the cone and cylinder be generated by the revolution of rt. △ ABC and rect. $CDEF$ about AC as an axis.)



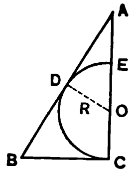
50. Find the area of the surface and the volume of a sphere circumscribing a cylinder of revolution, the radius of whose base is 9, and whose altitude is 24.

51. An equilateral triangle, whose side is 6, revolves about one of its sides as an axis. Find the area of the entire surface, and the volume, of the solid generated.

52. A cone of revolution is inscribed in a sphere whose diameter is $\frac{4}{3}$ the altitude of the cone. Prove that its lateral surface and volume are, respectively, $\frac{3}{8}$ and $\frac{9}{32}$ the surface and volume of the sphere.

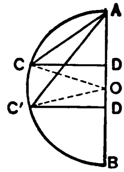
53. Find the volume of a sphere circumscribing a cube whose volume is 64.

54. A cone of revolution is circumscribed about a sphere whose diameter is two-thirds the altitude of the cone. Prove that its lateral surface and volume are, respectively, three-halves and nine-fourths the surface and volume of the sphere.



55. If the radius of a sphere is 25, find the lateral area and volume of an inscribed cone, the radius of whose base is 24.

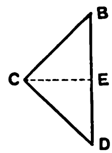
(Two solutions.)



56. If the volume of a sphere is $\frac{500}{3}\pi$, find the lateral area and volume of a circumscribed cone whose altitude is 18.

57. Find the volume of a spherical segment of one base whose altitude is 6, the diameter of the sphere being 30.

58. A square whose area is A revolves about its diagonal as an axis. Find the area of the entire surface, and the volume, of the solid generated.



59. The altitude of a cone of revolution is 9. At what distances from the vertex must it be cut by planes parallel to its base, in order that it may be divided into three equivalent parts? (§ 656.)

(Let V denote the volume of the cone, x the distance from the vertex to the nearer plane, and y the distance to the other.)

60. Given the radius of the base, R, and the total area, T, of a cylinder of revolution, to find its volume.

(Find H from the equation $T = 2\pi RH + 2\pi R^2$.)

61. Given the diameter of the base, D, and the volume, V, of a cylinder of revolution, to find its lateral area and total area.

62. Given the altitude, H, and the volume, V, of a cone of revolution, to find its lateral area.

63. Given the slant height, L, and the lateral area, S, of a cone of revolution, to find its volume.

64. A circular sector whose central angle is 45° and radius 12 revolves about a diameter perpendicular to one of its bounding radii. Find the volume of the spherical sector generated.

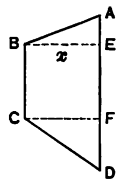
65. Given the area of the surface of a sphere, S, to find its volume.

66. Given the volume of a sphere, V, to find the area of its surface.

67. A right triangle, whose legs are a and b, revolves about its hypotenuse as an axis. Find the area of the entire surface, and the volume, of the solid generated.

68. The parallel sides of a trapezoid are 12 and 26, respectively, and its non-parallel sides are 13 and 15. Find the volume generated by the revolution of the trapezoid about its longest side as an axis.

(Represent BE by x.)



69. An equilateral triangle, whose altitude is h, revolves about one of its altitudes as an axis. Find the area of the surface, and the volume, of the solids generated by the triangle, and by its inscribed circle. (Ex. 21, p. 151.)

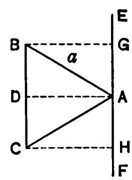
70. Find the lateral area and volume of a cylinder of revolution, whose altitude is equal to the diameter of its base, inscribed in a cone of revolution whose altitude is h, and radius of base r.

(Represent altitude of cylinder by x.)

71. Find the lateral area and volume of a cylinder of revolution, whose altitude is equal to the diameter of its base, inscribed in a sphere whose radius is r.

72. An equilateral triangle, whose side is a, revolves about a straight line drawn through one of its vertices parallel to the opposite side. Find the area of the entire surface, and the volume, of the solid generated.

(The solid generated is the difference of the cylinder generated by $BCHG$, and the cones generated by ABG and ACH.)



73. The outer diameter of a spherical shell is 9 in., and its thickness is 1 in. What is its weight, if a cubic inch of the metal weighs $\frac{1}{4}$ lb.? ($\pi = 3.1416$.)

74. Find the diameter of a sphere in which the area of the surface and the volume are expressed by the same numbers.

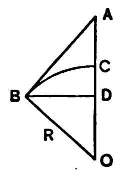
75. A regular hexagon, whose side is a, revolves about its longest diagonal as an axis. Find the area of the entire surface, and the volume, of the solid generated.

76. The sides AB and BC of rectangle $ABCD$ are 5 and 8, respectively. Find the volumes generated by the revolution of triangle ACD about sides AB and BC as axes.

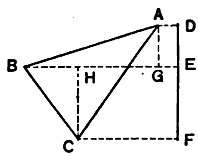
77. The sides of a triangle are 17, 25, and 28. Find the volume generated by the revolution of the triangle about its longest side as an axis. (§ 324.)

78. A frustum of a circular cone is equivalent to three cones, whose common altitude is the altitude of the frustum, and whose bases are the lower base, the upper base, and a mean proportional between the bases of the frustum. (§ 660.)

79. The volume of a cone of revolution is equal to the area of its generating triangle, multiplied by the circumference of a circle whose radius is the distance to the axis from the intersection of the medians of the triangle. (§ 140.)

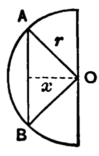


80. If the earth be regarded as a sphere whose radius is R, what is the area of the zone visible from a point whose height above the surface is H? (§ 271, 2.)



81. The sides AB and BC of acute-angled triangle ABC are $\sqrt{241}$ and 10, respectively. Find the volume of the solid generated by the revolution of the triangle about an axis in its plane, not intersecting its surface, whose distances from A, B, and C are 2, 17, and 11, respectively.

82. A projectile consists of two hemispheres, connected by a cylinder of revolution. If the altitude and diameter of the base of the cylinder are 8 in. and 7 in., respectively, find the number of cubic inches in the projectile. ($\pi = 3.1416$.)



83. A segment of a circle, whose bounding arc is a quadrant, and whose radius is r, revolves about a diameter parallel to its bounding chord. Find the area of the entire surface, and the volume, of the solid generated.

84. If any triangle be revolved about an axis in its plane, not parallel to its base, which passes through its vertex without intersecting its surface, the volume of the solid generated is equal to the area of the surface generated by the base, multiplied by one-third the altitude.

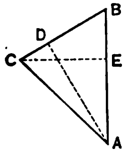


Fig. 1.

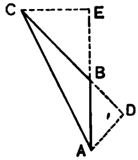


Fig. 2.

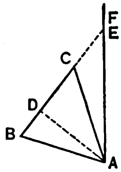


Fig. 3.

(Compare § 666. Case I., Figs. 1 and 2, when a side coincides with the axis; there are two cases according as AD falls on BC, or BC produced. Case II., Fig. 3, when no side coincides with the axis; prove by Case I.)

85. If any triangle be revolved about an axis which passes through its vertex parallel to its base, the volume of the solid generated is equal to the area of the surface generated by the base, multiplied by one-third the altitude.

(Compare Ex. 72, p. 383. There are two cases according as AD falls on BC, or BC produced.)

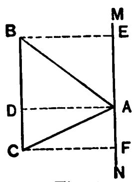


Fig. 1.

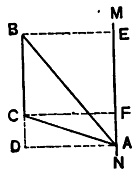
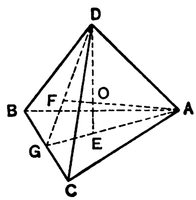


Fig. 2.

86. Find the area of the surface of the sphere circumscribing a regular tetraedron, whose edge is 8.

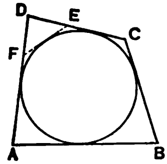
(Draw lines DOE and $AOF \perp$ to $\triangle$ ABC and BCD, respectively.)



APPENDIX.

PROOF OF STATEMENT MADE IN ELEVENTH LINE, PAGE 201.

683. Theorem. *The circumference of a circle is shorter than the perimeter of any circumscribed polygon.*



Given polygon $ABCD$ circumscribed about a ⊙.

To Prove circumference of ⊙ shorter than perimeter $ABCD$.

Proof. Of the perimeters of the ⊙ and of its circumscribed polygons, there must be one perimeter such that all the others are of equal or greater length.

But no circumscribed polygon can have this perimeter.

For, if we suppose polygon $ABCD$ to have this perimeter, and draw a tangent to the ⊙, meeting CD and DA at points E and F, respectively, then since str. line EF is $<$ broken line EDF, the perimeter of circumscribed polygon $ABCEF$ is $<$ perimeter $ABCD$.

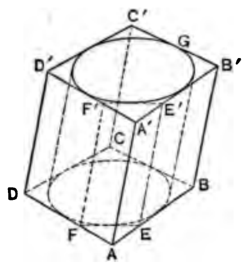
Hence, the circumference of the ⊙ is $<$ the perimeter of any circumscribed polygon.

PROOFS OF THE LIMIT STATEMENTS OF § 640.

684. We assume the following:

A portion of a plane is less than any other surface having the same boundaries.

685. Theorem. *The total surface of a circular cylinder is less than the total surface of any circumscribed prism.*



Given prism AC' circumscribed about circular cylinder EG.

To Prove total surface $EG <$ total surface AC'.

Proof. Of the total surfaces of the cylinder and of its circumscribed prisms, there must be one total surface such that the area of every other is either equal to or $>$ it.

But no circumscribed prism can have this total surface.

For suppose prism AC' to have this total surface; and let $BCDFE - E'$ be a circumscribed prism, whose face EF' intersects faces AB' and AD' in lines EE' and FF', respectively.

Now, face EF' is $<$ sum of faces AE', AF', AEF, and $A'E'F'$. (§ 684)

Whence, total surface of prism $BCDFE - E'$ is $<$ total surface of prism AC'.

Then, total surface of cylinder EG is $<$ total surface of any circumscribed prism.

Proofs of the Limit Statements of § 640.

686. Let L denote the lateral edge, H the altitude, S and s the the lateral areas, V and v the volumes, E and e the perimeters of rt. sections, and B and b the areas of the bases of the circumscribed and inscribed prisms, respectively; also, S' the lateral area of the cylinder, V' its volume, E' the perimeter of a rt. section, and B' the area of the base.

1. We have, $S + 2B > S' + 2B'$. (§ 685)

$$\therefore S + 2(B - B') > S'.$$

Again, the total surface of the inscribed prism is $<$ the total surface of the cylinder. (§ 684)

$$\therefore S' + 2B' > s + 2b, \text{ or } S' > s + 2(b - B').$$

Then, $S + 2(B - B') > S' > s + 2(b - B')$.

Now if the number of faces of the prisms be indefinitely increased, $B - B'$ and $b - B'$ approach the limit 0. (§ 363, II)

Again, the difference between the perimeters of the bases of the prisms approaches the limit 0. (§ 363, I)

Then, the total surface of the circumscribed prism continually decreases, but never reaches the total surface of the inscribed prism; and the total surface of the inscribed prism continually increases, but never reaches the total surface of the circumscribed prism. (§ 684)

Then, the difference between $S + 2B$ and $s + 2b$ can be made less than any assigned value, however small.

Whence, $S + 2B - (s + 2b)$, or $S - s + 2(B - b)$, approaches the limit 0.

But $B - b$ approaches the limit 0. (§ 363, II)

Whence, $S - s$ approaches the limit 0.

Then, S' is intermediate in value between two variables, the difference between which approaches the limit 0.

Then, the difference between either variable and S', that is,

$$S + 2(B - B') - S' \text{ and } S' - s - 2(b - B'),$$

approaches the limit 0.

Whence, $S - S'$ and $S' - s$ approach the limit 0.

Hence, S and s approach the limit S'.

2. We have, $V = B \times H$ and $v = b \times H$. (§ 499)

Whence, $V - v = B \times H - b \times H = (B - b) \times H$.

Now if the number of faces of the prisms be indefinitely increased, $B - b$, and therefore $V - v$, approaches the limit 0. (§ 363, II)

But V' is evidently $> v$, and $< V$.

Then, $V - V'$ and $V' - v$ approach the limit 0.

Whence, V and v approach the limit V'.

3. We have, $S = E \times L$ and $s = e \times L$. (§ 484)

Then, $E = \frac{S}{L}$ and $e = \frac{s}{L}$; or, $E - e = \frac{S - s}{L}$.

Now if the number of faces of the prisms be indefinitely increased, $S - s$, and therefore $E - e$, approaches the limit 0. (§ 640, 1)

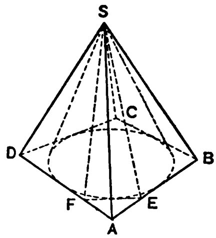
But E', the perimeter of a rt. section of the cylinder, is $< E$; for the theorem of § 683 is evidently true when for the ⊙ is taken any closed curve whose tangents do not intersect its surface; also, E' is $> e$. (Ax. 4)

Then, $E - E'$ and $E' - e$ approach the limit 0.

Whence, E and e approach the limit E'.

PROOFS OF THE LIMIT STATEMENTS OF § 650.

687. Theorem. *The total surface of a circular cone is less than the total surface of any circumscribed pyramid.*



Given pyramid $S\text{-}ABCD$ circumscribed about circular cone $S\text{-}EF$.

To Prove total surface $S\text{-}EF <$ total surface $S\text{-}ABCD$.

Proof. Of the total surfaces of the cone and of its circumscribed pyramids, there must be one total surface such that the area of every other is either equal to or $>$ it.

But no circumscribed pyramid can have this total surface.

For suppose pyramid $S\text{-}ABCD$ to have this total surface; and let $S\text{-}BCDFE$ be a circumscribed pyramid, whose face SEF intersects faces SAB and SAD in lines SE and SF, respectively.

Now, face SEF is $<$ sum of faces SAE, SAF, and AEF. (§ 684)

Whence, total surface of pyramid $S\text{-}BCDFE$ is $<$ total surface of pyramid $S\text{-}ABCD$.

Then, total surface of cone $S\text{-}EF$ is $<$ total surface of any circumscribed pyramid.

Proofs of the Limit Statements of § 650.

688. Let H denote the altitude, S and s the lateral areas, V and v the volumes, and B and b the areas of the bases, of the circumscribed and inscribed pyramids, respectively; also, S' the lateral area of the cone, V' its volume, and B' the area of its base.

1. We have, $S + B > S' + B'$. (§ 687)

$$\therefore\ S + (B - B') > S'.$$

Again, the total surface of the inscribed pyramid is $<$ the total surface of the cone. (§ 684)

$$\therefore\ S' + B' > s + b, \text{ or } S' > s + (b - B').$$

Then, $S + (B - B') > S' > s + (b - B')$.

Now if the number of faces of the pyramids be indefinitely increased, $B - B'$ and $b - B'$ approach the limit 0. (§ 363, II)

Also, the difference between the perimeters of the bases of the pyramids approaches the limit 0. (§ 363, I)

Then, $S + B$ continually decreases, and $s + b$ continually increases; and the difference between them can be made less than any assigned value, however small. (§ 684)

Then, $S - s + (B - b)$ approaches the limit 0.

But $B - b$ approaches the limit 0. (§ 363, II)

Whence, $S - s$ approaches the limit 0.

Then, S' is intermediate in value between two variables, the difference between which approaches the limit 0.

Whence, the difference between either variable and S', that is, $S + (B - B') - S'$ and $S' - s - (b - B')$, approaches the limit 0.

Then, $S - S'$ and $S' - s$ approach the limit 0.

Whence, S and s approach the limit S'.

2. We have, $V = B \times \frac{1}{3} H$ and $v = b \times \frac{1}{3} H$. (§ 521)

Whence, $V - v = (B - b) \times \frac{1}{3} H$.

Now if the number of faces of the pyramids be indefinitely increased, $B - b$, and therefore $V - v$, approaches the limit 0. (§ 363, II)

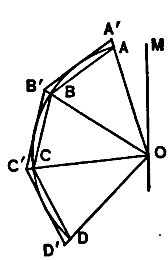
But, V' is evidently $> v$, and $< V$.

Then, $V - V$ and $V' - v$ approach the limit 0.

Whence, V and v approach the limit V'.

PROOF OF THE LIMIT STATEMENT IN NOTE FOOT OF PAGE 374.

689. Theorem. *If a regular broken line, inscribed in an arc, be revolved about a diameter, not intersecting the arc, as an axis, and the subdivisions of the arc be bisected indefinitely, the area of the surface generated by the broken line approaches the area of the surface generated by the arc as a limit.*



Given regular broken line $ABCD$, inscribed in arc AD, revolving about diameter OM as an axis.

To Prove that, if the subdivisions of arc AD be bisected indefinitely, area of surface generated by $ABCD$ approaches area of surface generated by arc AD as a limit.

Proof. Let $A'B'$, $B'C'$, and $C'D'$ be tangents ∥ to AB, BC, and CD, respectively, points A', B', C', and D' being in radii OA, OB, OC, and OD, respectively, produced; and let S, s, and S' denote the areas of the surfaces generated by $A'B'C'D'$, and $ABCD$, and arc AD, respectively.

Of the surfaces generated by arc AD, by $ABCD$, and by regular inscribed broken lines obtained by bisecting the subdivisions of the arc indefinitely, there must be one surface such that the areas of all the others are either equal to or $<$ it.

But no regular inscribed broken line can generate this surface.

For if this were the case, by bisecting the subdivisions of the arc, a regular inscribed broken line would be obtained having the same projection on the axis; but the ⊥ from O to each line would be greater, and hence the surface generated would be greater.

(§ 665, and Note foot of p. 374.)

Hence, surface generated by arc AD is $>$ surface generated by $ABCD$; that is, S' is $> s$.

Again, of the surfaces generated by arc AD, by $A'B'C'D'$, and by regular circumscribed broken lines obtained by bisecting the subdivisions of the arc indefinitely, there must be one surface such that the areas of all the others are either equal to or $>$ it.

But no regular circumscribed broken line can generate this surface.

For if this were the case, by bisecting the subdivisions of the arc, a regular circumscribed broken line would be obtained in which the ⊥ from O to each line would be the same; but the projection on the axis would be smaller, and hence the surface generated would be smaller.

Hence, surface generated by arc AD is $<$ surface generated by $A'B'C'D'$; that is, S' is $< S$.

Then, $S - S'$ and $S' - s$ are $< S - s$.

Now if the subdivisions of arc AD be bisected indefinitely, the difference between broken lines $A'B'C'D'$ and $ABCD$ approaches the limit 0. (Note foot p. 374.)

Then, the difference between the projections on OM of $A'B'C'D'$ and $ABCD$ approaches the limit 0.

Also, the difference between the ⊥s from O to $A'B'$ and AB approaches the limit 0. (Note foot p. 374.)

Then, the difference between the areas of the surfaces generated by $A'B'C'D'$ and $ABCD$, that is, $S - s$ approaches the limit 0. (§ 665)

Then, $S - S'$ and $S' - s$ approach the limit 0.

Whence, S and s approach the limit S'.

PROOF OF THE LIMIT STATEMENT IN NOTE FOOT OF PAGE 377.

690. Theorem. *If a regular polygonal sector, inscribed in a sector of a circle, be revolved about a diameter, not crossing the sector, as an axis, and the subdivisions of the arc be bisected indefinitely, the volume of the solid generated by the polygonal sector approaches the volume of the solid generated by the sector as a limit.*

Given regular polygonal sector $OABCD$, inscribed in sector OAD, revolved about diameter OM as an axis. (Fig. of § 689.)

To Prove that, if the subdivisions of arc AD be bisected indefinitely, volume of solid generated by $OABCD$ approaches volume of solid generated by sector OAD as a limit.

Proof. Let $A'B'$, $B'C'$, and $C'D'$ be tangents ∥ to AB, BC, and CD, respectively, points A', B', C', and D' being in radii OA, OB, OC, and OD, respectively, produced; and let V, v, and V' denote the volumes of the solids generated by $OA'B'C'D'$, $OABCD$, and sector OAD, respectively.

Then, V' is evidently $> v$, and $< V$.

Whence, $V - V'$ and $V' - v$ are $< V - v$.

Now if the subdivisions of arc AD be bisected indefinitely, the difference between the areas of $OA'B'C'D'$ and $OABCD$, and therefore $V - v$, approaches the limit 0. (Note foot p. 377.)

Then, $V - V'$ and $V' - v$ approach the limit 0.

Whence, V and v approach the limit V'.

ANSWERS

TO

NUMERICAL EXERCISES.

Book I.

4. $24°$. **5.** $63° 30'$, $26° 30'$. **8.** $22° 30'$, $157° 30'$.
9. $37°$. **24.** $A = 112° 30'$, $B = C = 33° 45'$. **88.** 7.

Book II.

12. $28°$. **13.** $44° 30'$. **14.** $12°$. **15.** $54° 30'$. **16.** $178°$.
17. $112° 30'$. **18.** $83°$, $89° 30'$, $97°$, $90° 30'$, $74° 30'$.
52. $\angle AED = 14° 30'$, $\angle AFB = 10° 30'$.
55. $114° 30'$, $89° 30'$, $65° 30'$, $90° 30'$.
67. $97° 30'$, $89° 30'$, $82° 30'$, $90° 30'$.

Book III.

1. 112. **2.** 42. **3.** $\frac{25}{27}$. **4.** 63. **5.** BC, $3\frac{1}{5}$, $2\frac{4}{5}$; CA, 4, 3; AB, $4\frac{4}{13}$, $3\frac{9}{13}$. **6.** BC, $11\frac{2}{3}$, $18\frac{2}{3}$; CA, 20, 28; AB, 35, 40.
7. $19\frac{3}{5}$, $25\frac{1}{5}$. **9.** 4 ft. 6 in. **10.** 12. **11.** 15.
12. 37 ft. 1 in. **13.** 47 ft. 6 in. **14.** $\frac{10}{3}\sqrt{3}$.
15. $15\sqrt{2}$ in. **16.** 41. **17.** 58. **18.** 21. **19.** 24.
25. 18. **28.** 48. **29.** 10. **30.** $13\frac{1}{3}$. **31.** $9\sqrt{2}$. **32.** 45.
34. $17\frac{2}{3}$. **37.** 50. **41.** $\sqrt{129}$, $2\sqrt{21}$, $\sqrt{201}$. **42.** $\frac{10}{3}$.
47. 36. **49.** 63. **50.** 4 and 3; $\frac{16}{5}$ and $\frac{9}{5}$. **56.** 24.

57. 17. **58**. 21, 28. **59**. $8\sqrt{3}$. **60**. $BE = 4$, $ED = 12$.
62. $6\sqrt{3}$. **67**. 14. **70**. 21. **74**. $\frac{99}{18}$ and $\frac{70}{18}$; 9 and 5.

Book IV.

1. $30\frac{5}{6}$ ft. **2**. 8 ft. 9 in. **3**. 14, 12. **4**. 6 ft. 11 in., 20 ft. 9 in. **5**. 6 sq. ft. 60 sq. in. **6**. $30\sqrt{3}$. **7**. 26 yd. 1 ft. **8**. 2 sq. ft. 48 sq. in. **9**. 243. **10**. 210; $24\frac{12}{17}$, 15, $16\frac{4}{5}$. **11**. 73. **12**. 117. **16**. 2 ft. 10 in. **18**. $\frac{25}{4}\sqrt{3}$. **19**. $3\sqrt{3}$. **21**. 120. **24**. 210. **25**. 18. **26**. $1\frac{1}{2}$ ft. **27**. 6. **28**. $4\sqrt{3}$. **29**. 1260. **33**. 150. **34**. 17. **36**. 624. **37**. 540 sq. in. **38**. 28 ft. **41**. $\frac{13}{8}$. **42**. 30, 16. **45**. $\frac{145}{4}$. **47**. $AD = \frac{15}{2}\sqrt{2}$, $AE = 11\sqrt{2}$. **48**. 54. **51**. Area $ABD = 39$, area $ACD = 45$. **52**. 1010. **53**. 336.

Book V.

32. Area, $\frac{625}{4}\pi$. **33**. Circumference, 34π. **34**. 64 : 121. **36**. 9. **37**. 13. **38**. $\frac{9}{2}\sqrt{2}$. **39**. $\frac{34}{3}\pi$. **40**. $\frac{57}{5}\pi$. **41**. 9.8268. **42**. $\frac{13}{4}\pi$. **43**. 392. **44**. 48π. **45**. 1.2732. **46**. $\frac{64}{9}\pi$. **47**. 6π. **48**. 16π. **49**. 3π, 12π. **50**. 8π, $8\pi\sqrt{2}$. **51**. 9.06. **52**. 416π sq. ft. **53**. 120.99 ft. **54**. 57 in. **60**. 57.295°+. **61**. 2.658+. **62**. 5.64+.

APPENDIX TO PLANE GEOMETRY.

58. $10\sqrt{7}$. **62**. 8. **63**. $\frac{29}{16}$. **91**. 480.

Book VII.

1. 4 : 3. **2**. 2 : 5. **4**. 42. **5**. 1 ft. 9 in. **6**. $34\frac{21}{64}$ cu. in.; $63\frac{3}{8}$ sq. in. **7**. 574. **8**. 1008. **9**. 12, 7. **10**. 1944. **12**. Volume, $50\sqrt{3}$. **14**. Volume, $\frac{243}{2}\sqrt{3}$. **15**. 17. **17**. 2400 sq. in. **18**. 770. **19**. Volume, $48\sqrt{5}$. **20**. 144. **21**. 512, 384. **22**. 1705. **23**. 10, 1. **24**. 36 sq. in. **25**. 12 in. **28**. $\sqrt{273}$, $18\sqrt{237}$, $180\sqrt{3}$. **29**. $\frac{1}{2}\sqrt{118}$,

$3\sqrt{109}$, 15. **30**. $\sqrt{97}$, $12\sqrt{93}$, $72\sqrt{3}$. **31**. $4\sqrt{39}$, $504\sqrt{3}$, $936\sqrt{3}$. **32**. $6\sqrt{3}$, $56\sqrt{26}$, $503\frac{1}{3}$. **33**. $4\sqrt{10}$, $72\sqrt{39}$, $672\sqrt{3}$. **34**. 150. **35**. 320. **36**. 840. **37**. 700, 1568. **38**. $\frac{7}{3}\sqrt{57}$, $640\sqrt{3}$. **39**. $42\sqrt{91}$, $624\sqrt{3}$. **40**. 108, $21\sqrt{39}$. **41**. 240, $\frac{100}{3}\sqrt{119}$. **49**. $4\sqrt{3}$, $\frac{2}{3}\sqrt{2}$. **50**. 15. **51**. 768, 2340. **59**. 438. **63**. 9600 lb. **64**. 50. **69**. $168\sqrt{3}$, $15\sqrt{219}$. **76**. 3456 cu. in. **77**. 6 ft. **78**. 4 ft. 6 in. **79**. $5\sqrt[3]{4}$ in. **80**. 960, 3072. **81**. 128. **82**. 12. **83**. 6. **86**. $36\sqrt{3}$.

Book VIII.

7. $39\frac{2}{3}$. **9**. 86° 24′. **10**. 36. **11**. 44. **12**. 3 : 2. **13**. 108°. **14**. 220. **16**. $33\frac{4}{5}$. **17**. $66\frac{1}{2}$. **18**. $36\frac{1}{4}$. **19**. 60. **20**. 153°. **23**. 45°. **24**. $4\frac{4}{5}$. **26**. $4\sqrt{3}$, $\sqrt{3}$. **44**. 30, 8, 20.

Book IX.

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